EXAM 2
Math 2513
6-13-14

Name key

1. *(15 points)* Suppose a and b are positive integers.

a) Prove that if 7 divides $16b$, then 7 divides b. (Hint: use the fundamental theorem of arithmetic.)

[10] By the fundamental theorem, b can be written as a product of primes, and there is only one way to do this (apart from changing the order of the primes). 3 Then $16b$ can be written as 2^4 times the primes 3 which are factors of b. Since 7 divides $16b$, then 7 is one of the prime factors of $16b$, and since $7 \neq 2$, then 7 must be one of the prime factors of b. So 7 divides b.

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[5] b) Prove, by counterexample, that the statement "if 14 divides $16b$, then 14 divides b" is false.

Let $b = 7$. Then $16b = 16 \cdot 7 = 8 \cdot 2 \cdot 7 = 8 \cdot 14$, so 14 divides $16b$. But $14 \nmid 7$ (because any number which divides 7 must be less than or equal to 7). So $14 \nmid b$.

2. *(10 points)* Decide whether the following statement is true or false. If it is true, give a proof. If it is false, give a counterexample.

For all real numbers a and b, $|a| - |b| = |a - b|$.

The statement is false. Let $a = 1$ and $b = -1$. Then

$|a| - |b| = |1| - |-1| = 1 - 1 = 0$, but

$|a - b| = |1 - (-1)| = |2| = 2$. So there exist a and b such that $|a| - |b| \neq |a - b|$.

3. *(15 points)* Suppose x is a real number. Prove that if x is not an integer, then

$$\lfloor -x \rfloor = -\lfloor x \rfloor - 1.$$

Suppose $x \in \mathbb{R}$ and x is not an integer.
Let $n = \lfloor x \rfloor$. By definition, n is an integer such that $n \leq x < n + 1$. 4 Then by the properties of inequalities, it follows that $-n \geq -x > -(n+1)$. So

$$-n \geq -x > -n - 1, \text{ or}$$

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$$-n - 1 < -x \leq -n.$$

Since $-n-1 < -x$, then $-n - 1 \leq -x$, and since x is not an integer, $x \neq -n$, therefore $-x \leq -n$ implies $-x < -n$. So

3 $-n - 1 \leq -x < -n$. Hence, by definition, $\lfloor -x \rfloor = -n - 1$.

4 So $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$.

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4. *(15 points)* Use the Euclidean algorithm to find the greatest common divisor of 427 and 154. Show all work!

(1) $154 \overline{)427}$ with quotient 2; $427 - 308 = 119$, so $427 = 2 \cdot 154 + 119$

(2) $119 \overline{)154}$ with quotient 1; $154 - 119 = 35$, so $154 = 1 \cdot 119 + 35$

(3) $35 \overline{)119}$ with quotient 3; $119 - 105 = 14$, so $119 = 3 \cdot 35 + 14$

[10]

(4) $14 \overline{)35}$ with quotient 2; $35 - 28 = 7$, so $35 = 2 \cdot 14 + 7$

(5) $14 = 2 \cdot 7 + 0$.

[5] → Since the remainder at the last step is zero, then 7 is the greatest common divisor of 427 and 154.

5. *(15 points)* Prove the following statement by proving its contrapositive:

For all integers n, if $n \textit{ mod } 4 = 2$ or $n \textit{ mod } 4 = 3$, then n is not a perfect square.

Suppose n is an integer and n is a perfect square. (2) Then there exists an integer k such that $n = k^2$. (2)

By the quotient-remainder theorem, either k is odd or k is even.

Case 1: Suppose k is even. Then there exists an integer l such that $k = 2l$. So $n = k^2 = (2l)^2 = 4l^2$. So n is divisible by 4, or $n \bmod 4 = 0$. So $n \bmod 4 \neq 2$ and $n \bmod 4 \neq 3$. (5)

Case 2: Suppose k is odd. Then there exists an integer l such that $k = 2l + 1$. So $n = k^2 = (2l+1)^2 = 4l^2 + 4l + 1$. So $n = 4(l^2 + l) + 1$. Since $l^2 + l$ is an integer, this implies that $n \bmod 4 = 1$. So $n \bmod 4 \neq 2$ and $n \bmod 4 \neq 3$. (5)

Thus, in all cases, we've proved that if n is a perfect square then $n \bmod 4 \neq 2$ and $n \bmod 4 \neq 3$, which is the contrapositive of the desired conclusion. // (1)

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6. *(15 points)* Use induction to prove that for all integers $n \geq 3$,

$$n! \geq 3^{n-2}.$$

For $n \geq 3$,

Let $P(n)$ be the statement that $n! \geq 3^{n-2}$. 2

We'll prove $P(n)$ by induction for $n \geq 3$. 1

Basis step: $P(3)$ says that $3! \geq 3^{3-2}$, which is true because 2 $3! = 6$ and $3^{3-2} = 3^1 = 3$, and $6 \geq 3$.

Inductive step: Let $k \geq 3$ be an integer such that $P(k)$ is true, so $k! \geq 3^{k-2}$. 2

Then $(k+1)! = k!\,(k+1)$ 2 (by the ~~inductive~~ recursive definition of factorial)

$\leq 3^{k-2} \cdot (k+1)$ 2 (by the inductive hypothesis)

$\leq 3^{k-2} \cdot 3$ 2 (we have $k+1 \geq 3$ since $k \geq 3$)

$= 3^{k-1} = 3^{(k+1)-2}$. 2 So $(k+1)! \geq 3^{(k+1)-2}$, so $P(k+1)$ is true. //

7. *(15 points)* Use induction to prove that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

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For $n \geq 1$, let $P(n)$ be the statement that the desired equality holds for this value of n. We'll prove $P(n)$ by induction. 1

Basis step: When $n = 1$, we have $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$ and 2 $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$, so the desired equality holds, so $P(1)$ is true.

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Inductive step: Let k be an integer and suppose $P(k)$ is true.

Then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$ 2 (by the recursive definition of summation)

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ 2 (by the inductive hypothesis)

$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} =$

2

$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$. So $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(\cancel{k+1})(k+2)}$,

2 which means that $P(k+1)$ is true. //