EXAM 1
Math 2513
5-30-14

Name key

1. *(15 points)*

[9] a) Construct a truth table for the compound statements $(p \to q) \to r$ and $(\sim p) \vee q$. Include columns for p, q, r, $p \to q$, $(p \to q) \to r$, and $(\sim p) \vee q$.

premises both true

p	q	r	$p \to q$	$(p \to q) \to r$	$(\sim p) \vee q$	
T	T	T	T	T	T	☆
T	T	F	T	F	T	
T	F	T	F	T	F	
T	F	F	F	T	F	
F	T	T	T	T	T	☆
F	T	F	T	F	T	
F	F	T	T	T	T	☆
F	F	F	T	F	T	

3 points for each of the last three columns.

[6] b) Using the truth table you constructed above, explain whether the following is a valid argument:

$$(p \to q) \to r$$
$$(\sim p) \vee q$$
$$\therefore r$$

3 The premises $(p \to q) \to r$ and $(\sim p) \vee q$ are both true in lines 1, 5, and 7 of the truth table (marked with ☆). In all three of these lines, the conclusion r is also true. So the argument is valid. 3

2. *(15 points)* Use the table of logical equivalences provided with this test to prove that

$$(\sim ((\sim p) \vee q)) \vee (p \wedge q) \equiv p.$$

Supply a reason for each step of your proof.

$(\sim ((\sim p) \vee q)) \vee (p \wedge q) \equiv ((\sim\sim p) \wedge \sim q) \vee (p \wedge q)$ 3 (De Morgan's laws)

$\equiv (p \wedge \sim q) \vee (p \wedge q)$ 3 (Double negative law)

$\equiv p \wedge (\sim q \vee q)$ 3 (Distributive law)

$\equiv p \wedge t$ 3 (Negation law)

$\equiv p$ 3 (Identity law).

3. *(15 points)* You are on the island of knights and knaves, where knights always make true statements and knaves always makes false statements. You meet three inhabitants, A, B, and C.

A says: Exactly one of B and C is a knave.
B says: A is a knight.

Which is C, a knight or a knave? Prove your answer.

(2) Consider two cases: either B is a knight or B is a knave.

(6) If B is a knight, then since he is telling the truth, A must be a knight. Therefore A's statement is true, and since B is not a knave, it follows that C is a knave.

(6) If B is a knave, then since he is lying, A must be a knave. Therefore A's statement is false, and since B is a knave, it follows that C must also be a knave.

(1) So in either case, C must be a knave.

4. *(10 points)* Let $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{1, 2, 3\}$. For each of the following statements, decide whether the statement is true or false, and give a complete justification of your answer.

[5] a) $\exists x \in S\ \forall y \in T\ \ x = 2y$. This is false. We can prove this by showing that the negation $\forall x \in S\ \exists y \in T\ \ x \neq 2y$ is true. Let $x \in S$ be given. Then $x = 2y$ can be true for at most one element y of T, so $x = 2y$ can't be true for all $y \in T$, since there is more than one $y \in T$. So $x \neq 2y$ for some $y \in T$.

[5] b) $\forall y \in T\ \exists x \in S\ \ x = 2y$

This is true. Proof by cases: Let $y \in T$ be given. Either $y = 1$, $y = 2$, or $y = 3$. If $y = 1$, then $x = 2 \cdot y = 2 \in S$. If $y = 2$, then $x = 2y = 4 \in S$. If $y = 3$, then $x = 2y = 6 \in S$. So in all cases, $\exists x \in S$ such that $x = 2y$.

5. *(10 points)* Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. Consider the relation $R \subset A \times B$ given by

$$R = \{(1, 4), (2, 4), (3, 5), (4, 3), (4, 4), (5, \cancel{1})\}$$

5

[6] a) Is R a function with domain A? Explain your answer.

No, the statement "If $(x,y) \in R$ and $(x,z) \in R$ then $y = z$" is not true (3), because of the counterexample: $(4,3) \in R$ and $(4,4) \in R$ and $3 \neq 4$. So R does not have one of the properties needed to be a function. (3)

b) Remove one element from R so as to make it a function with domain A.

[4] If we remove $(4,3)$ from R (removing $(4,4)$ will also work) then each element of A does get assigned by R to just one element of B, so R does define a function.

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6. *(15 points)* Prove that if you take any three consecutive integers and multiply the largest by the smallest, the result is one less than a perfect square. (For example, if the three consecutive integers are 7, 8, and 9, then multiplying 7 by 9 gives 63, which is one less than a perfect square.)

Proof: Let any set of three consecutive integers be given, and let n denote the smallest of the three. 3 Then the three integers are n, $n+1$, and $n+2$; and the largest of the three is $n+2$. 3 Multiplying the largest by the smallest gives $n(n+2) = n^2+2n$. 3
This result is one less than n^2+2n+1 3, which is a perfect square since $n^2+2n+1 = (n+1)^2$.

3

7. *(10 points)* Decide whether the following statement is true or false, and prove your answer: If n is a positive even number, then n^2+1 is prime.

It is false, as is shown by the counterexample $n=8$. We have that $n=8$ is even, and $n^2+1 = 65 = 5\cdot 13$, so n^2+1 is not prime.

5 pts: giving an even number as a counterexample
5 pts: showing that n^2+1 is not prime for this number.

8. *(10 points)* Show that if p is a rational number and s is an irrational number, then $s-p$ is irrational. (Hint: assume that $s-p$ is rational and derive a contradiction.)

Proof: Suppose p is rational and s is irrational. 2 We have to show that $s-p$ is irrational. To prove this, we assume that $s-p$ is rational and get a contradiction. 2 If $s-p$ is rational, then since (as proved in class) the sum of two rational numbers is 2 rational, it follows that $(s-p)+p$ is rational. But $(s-p)+p = s$, 2 so then s is rational, contradicting the assumption that s is irrational. 2

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology **t** and a contradiction **c**, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$