EXAM 3
Math 2513
6-28-13

Name key

1. *(15 points)* Find all the possible numbers that can be formed by adding up 3's and 11's. Prove your answer using induction.

~~3~~ ~~6~~ ~~9~~ ~~12~~ ~~15~~ ~~18~~ ~~21~~ 24 27 30 33 36 ...
~~11~~ ~~14~~ ~~17~~ ~~20~~ ~~23~~ 26 29 32 ...
~~22~~ 25 28 31 34 ...

3, 6, 9, 11, 12, 14, 15, 17, 18, 20, 21, 22, 23, ... (5)

The answer is: 3, 6, 9, 11, 12, 14, 15, 17, 18, and all numbers ≥ 20.

Let $P(n)$ be the statement that n is a sum of 3's and 11's.
Clearly $P(n)$ is true for 3, 6, 9, 11, 12, 14, 15, 17, 18, 20, 21, and 22. (4)
Now suppose $P(n)$ is true for $20 \leq n \leq k$ and $k \geq 22$. (3)
Then $20 \leq k-2 \leq k$, so $P(k-2)$ is true.
So $k-2$ can be written as a sum of 3's and 11's. Then adding one more 3 gives $k+1$ as a sum of 3's and 11's. So $P(k+1)$ is true. (3)

2. *(15 points)*

a) How many ways are there to arrange the 9 letters AAABBBBCC?

[5] This is a Mississippi problem. Since there are 9 letters with 3 A's, 4 B's and 2 C's, the answer is $\dfrac{9!}{3!\,4!\,2!}$.

b) How many ways are there to arrange, without having two D's together, the 14 letters AAABBBBCCDDDDD?

Mark 9 spaces for the 9 letters which are not D's, and 10 wedges between them as places for the D's:

(3)

^ — ^ — ^ — ^ — ^ — ^ — ^ — ^ — ^ — ^

There are (2) $\binom{10}{5}$ ways to choose 5 of the wedges as places for the D's, and, by part a), there are $\dfrac{9!}{3!\,4!\,2!}$ (2) ways to arrange the remaining letters in the 9 spaces.

So the answer is $\binom{10}{5} \times \dfrac{9!}{3!\,4!\,2!}$ (3)

3. *(15 points)* Prove your answers using the pigeonhole principle.

[5] a) How many people must be in a room in order to be sure that at least three of them have the same birthday?

If $n = 2 \times 365 + 1$, then the least integer greater than or equal to $\frac{n}{365}$ is 3, so there must be at least 3 people with the same birthday by the generalized pigeonhole principle (The boxes are birthdays and the pigeons are people). So If $n = 2 \times 365$, it's possible for each birthday to have only two people. The answer is $2 \times 365 + 1$.

(5)

[10] b) There are 20 people in a room. Each of them has a certain number, possibly zero, of friends in the room (not including himself). Prove that there are two people in the room who have exactly the same number of friends in the room.

There are 20 boxes, numbered 0 through 19. Put each of the 20 people in the box with the number of that person's friends on it.

(3)

There are two cases: at least one box is empty, or all boxes are full.

If one box is empty, then there are twenty people in the remaining 19 boxes, so by the pigeonhole principle, at least one box contains two people, and then those two people have the same number of friends.

(3)

If no box is empty, then the box numbered 0 has a person in it, so one person has no friends. But the box numbered 19 has a person in it, so that person has everybody as a friend, including the person with no friends. That is a contradiction. So the second case can't occur.

(4)

4. *(12 points)*

a) How many 5-letter words can be formed from the alphabet without repeating any letter?

[6] This is the number of ways to choose 5 objects from 26, with order counting: it equals ~~[crossed out]~~ $26 \times 25 \times 24 \times 23 \times 22$.

b) How many ways are there to pair off 8 women with 8 men at a dance?

[6] Line up the 8 women. Then the men can be paired with the women by lining them up in order. There are $8!$ ways to do this.

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5. *(8 points)* How many non-negative numbers less than a billion have five 7's?

Each non-negative number less than a billion is a string of 9 digits, where each digit can be any number from 0 through 9. (2)
There are $\binom{9}{5}$ ways to choose which 5 of the 9 digits will be 7's. (2)
There are 9^4 ways to fill in the remaining 4 digits with one of the nine numbers $\{0,1,2,3,4,5,6,8,9\}$. (3)
So the answer is $\binom{9}{5} \times 9^4$. (1)

6. *(15 points)* Let $A = \{1, 2, 3, 4, 5\}$ and

$$R = \{(1,1), (1,2), (2,3), (3,2), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}.$$

Answer the following three questions, with proof.

[3] a) Is R reflexive?
Yes, because (1,1), (2,2), (3,3), (4,4), and (5,5) are all in R

[3] b) Is R symmetric?
No, because $(1,2) \in R$ but $(2,1) \notin R$

[3] c) Is R transitive?
No, because (1,2) and (2,3) are in R but $(1,3) \notin R$.

[3] d) Add three pairs to R which make it an equivalence relation.
Add (2,1), (1,3), and (3,1) to R. Then R is reflexive, symmetric, and transitive, so it is an equivalence relation.

[3] e) What are the equivalence classes of the equivalence relation in part d)?
$\{1,2,3\}$ and $\{4,5\}$.

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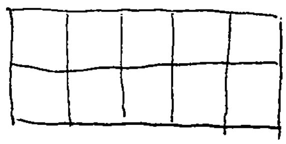
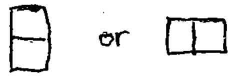
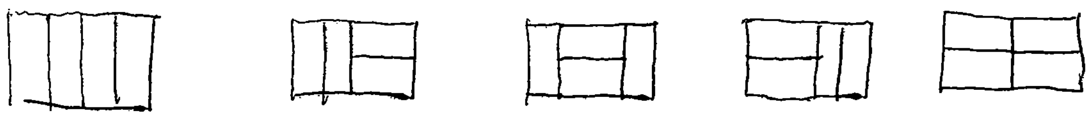
7. *(20 points)* Consider a $2 \times n$ checkerboard, with 2 rows of n squares each, such as this board with $n = 5$:



Let a_n be the number of different ways to cover the $2 \times n$ checkerboard with dominos. By a domino, we mean two squares joined together:



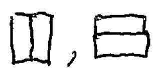
For example, $a_4 = 5$, because there are five different ways to cover a 2×4 board with dominos:



[10] a) Find a_1, a_2, a_3, and a_5 (draw pictures in support of your answer).

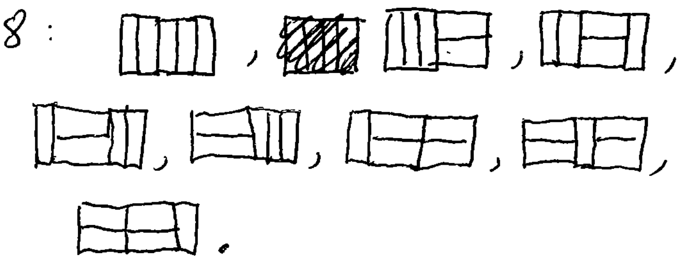
(2) $a_1 = 1$:

(2) $a_2 = 2$:



(2) $a_3 = 3$:

(4) $a_5 = 8$:



[10] b) Can you guess a relation between a_n and another sequence that you've seen before? Use induction to prove that your guess is correct.

Let b_n be the Fibonacci numbers: $b_1 = 1, b_2 = 1, b_3 = 2, b_4 = 3, \ldots$ where $b_{n+1} = b_n + b_{n-1}$ for all $n \geq 2$.

We claim that $a_n = b_{n+1}$ for all $n \geq 1$. (2)

To prove this by induction, let $P(n)$ be the statement $a_n = b_{n+1}$. As seen in part a), $P(n)$ is true for $n = 2$ (since $a_2 = 2 = b_3$) and for $n = 1$ (since $a_1 = 1 = b_2$) (2)

Now assume $P(n)$ is true for all n such that $1 \leq n \leq k$, where $k \geq 2$. Then since $1 \leq k-1 \leq 2$, both $P(k)$ and $P(k-1)$ are true.

(2) Each cover of a $2 \times (k+1)$ board by dominos must end with either two horizontal or one vertical domino. The number ending with two horizontal dominos is the same as the number of covers of a $2 \times (k-1)$ board, which is a_{k-1}. The number ending with one vertical domino is the same as the number of covers of a $2 \times (k)$ board, which is a_k. (2)

(2) By inductive hypotheses, $a_{k-1} = b_k$ and $a_k = b_{k+1}$. So $a_{k+1} = b_k + b_{k+1} = b_{k+2}$

So $P(k+1)$ is true.