EXAM 3
Math 2513
11-16-12

Name key

1. *(12 points)* Use the Euclidean algorithm to find the greatest common divisor of 315 and 417.

(3) $417 = 1 \cdot 315 + 102$

(3) $315 = 3 \cdot 102 + 9$

(3) $102 = 11 \cdot 9 + \boxed{3}$ ← so the greatest common divisor is $\boxed{3}$ (3)

$9 = 3 \cdot 3 + 0$

2. *(12 points)* Use the computations from problem 1 above to express the greatest common divisor of 315 and 417 as a linear combination of 315 and 417.

$3 = 102 - 11 \cdot 9$ (2)

Using $9 = 315 - 3 \cdot 102$ gives (2)

$3 = 102 - 11 \cdot (315 - 3 \cdot 102)$ (2)

or $3 = 102 - 11 \cdot 315 + 33 \cdot 102$

or $3 = 102 \cdot 34 - 315 \cdot 11$ (2)

Using $102 = 417 - 315$ gives

$3 = (417 - 315) \cdot 34 - 315 \cdot 11$ (2)

$3 = 417 \cdot 34 - 315 \cdot 34 - 315 \cdot 11$

$\boxed{3 = 417 \cdot 34 - 315 \cdot 45}$ (2)

3. *(8 points)* If the product of two integers is $2^{11} \cdot 3^{12} \cdot 11 \cdot 17^6$ and their least common multiple is $2^8 \cdot 3^7 \cdot 11 \cdot 17^3$, what is the greatest common divisor? (Explain your answer.)

Call the integers m and n. Then $\gcd(m,n) = \dfrac{m \cdot n}{\operatorname{lcm}(m,n)}$ (4)

$$\text{So } \gcd(m,n) = \frac{2^{11} \cdot 3^{12} \cdot 11 \cdot 17^6}{2^8 \cdot 3^7 \cdot 11 \cdot 17^3} \ (2) = \boxed{2^3 \cdot 3^5 \cdot 17^3} \ (2)$$

4. *(15 points)* Prove that 3 divides $n^3 - 4n$ whenever n is a positive integer.

Let $P(n)$ be the statement "$3 \mid (n^3-4n)$". (2)

For $n=1$ we have $n^3 - 4n = 1 - 4 = -3$, which is divisible by 3, so $P(1)$ is true. (3)

Now assume $P(n)$ is true. (2) Then there exists $k \in \mathbb{Z}$ such that $n^3 - 4n = 3k$. (2) We have

$$\begin{aligned}
(n+1)^3 - 4(n+1) &= (n^3 + 3n^2 + 3n + 1) - (4n+4) \quad (2)\\
&= n^3 + 3n^2 - n - 3\\
&= (n^3 - 4n) + 4n + 3n^2 - n - 3 \quad (2)\\
&= \underbrace{(n^3-4n)}_{\text{by the inductive hypothesis}} + 3n + 3n^2 - 3 = 3k + 3n + 3n^2 - 3\\
&= 3(k + n + n^2 - 1). \quad (2)
\end{aligned}$$

Since $(k+n+n^2-1) \in \mathbb{Z}$, then $3 \mid [(n+1)^3 - 4(n+1)]$, so $P(n+1)$ is true.

5. *(15 points)* Recall that the Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Use induction to prove that for every positive integer n,

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

(2) Let $P(n)$ be the statement that $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$.

For $n=1$ we have $F_1 = 1$ and $F_{n+2} = F_3 = F_1 + F_2 = 1 + 1 = 2$,

so $F_1 = F_{n+2} - 1$ is true, so $P(1)$ is true. (3)

Now assume $P(n)$ is true.

$$\begin{aligned}
\text{Then } F_1 + F_2 + \ldots + F_{n+1} &= (F_1 + F_2 + \ldots + F_n) + F_{n+1} \quad (3)\\
&= (F_{n+2} - 1) + F_{n+1} \quad (3) \quad \text{(by the inductive hypotheses)}\\
&= (F_{n+2} + F_{n+1}) - 1\\
&= F_{n+3} - 1 \quad (3) \quad \text{(by the definition of the Fibonacci sequence)}
\end{aligned}$$

This shows that $P(n+1)$ is true, (1)

so the statement is proved by induction.

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6. *(38 points)* Answer the following questions, giving reasons for your answers.

[16]

a) How many strings of 8 uppercase letters start with OK ?

$$\underset{O}{\underline{1}} \times \underset{K}{\underline{1}} \times \underbrace{\underline{26} \times \underline{26} \times \underline{26} \times \underline{26} \times \underline{26} \times \underline{26}}_{\text{any letters}} = 26^6$$

b) How many strings of 8 uppercase letters end with HOMA ?

~~H O M A~~

$$\underbrace{\underline{26} \times \underline{26} \times \underline{26} \times \underline{26}}_{\text{any letters}} \times \underset{H}{\underline{1}} \times \underset{O}{\underline{1}} \times \underset{M}{\underline{1}} \times \underset{A}{\underline{1}} = 26^4$$

c) How many strings of 8 uppercase letters start with OK and end with HOMA ?

$$\underset{O}{\underline{1}} \times \underset{K}{\underline{1}} \times \underbrace{\underline{26} \times \underline{26}}_{\text{any letters}} \times \underset{H}{\underline{1}} \times \underset{O}{\underline{1}} \times \underset{M}{\underline{1}} \times \underset{A}{\underline{1}} = 26^2$$

[8]

d) How many strings of 8 uppercase letters start with OK or end with HOMA?

From a) we know there are 26^6 strings starting with OK, and from b) 4 we know there are 26^4 strings ~~starting~~ ending with HOMA. If we add these numbers, we count all strings starting with OK or ending with HOMA, but we count all the strings in part c) twice. 4 So we have to subtract off the number counted twice. Answer: $\boxed{26^6+26^4-26^2}$

[6]

e) How many strings of 8 uppercase letters start with OK and do not end with HOMA?

3 This is equal to the number of strings starting with OK (counted as 26^6 in a)) minus the number of strings starting with OK and ending with HOMA 3 (counted as 26^2 in c)).

So the answer is $\boxed{26^6-26^2}$.

[8]

f) How many strings of 8 uppercase letters do not start with OK and do not end with HOMA?

If w is any string of 8 uppercase letters, then the statement

"w does not start with OK and w does not end with HOMA"

is logically equivalent to the statement

4 "it is not true that [w ~~???~~ starts with OK or w ends with HOMA]".

So the number of strings not starting 2 with OK and not ending with HOMA is (total no. of strings of 8 letters) − (no. starting with OK or ending with HOMA).

Therefore, by part d), the answer is 2 $\boxed{26^8-[26^6+26^4-26^2]}$.