EXAM 2
Math 2513
10-19-12

Name key

1. *(20 points)* Suppose $f : [-1, \infty) \to [0, \infty)$ is defined by $f(x) = \sqrt{x+1}$.

a) Prove that f is a bijection.

[8] Suppose $x_1 \in [-1,\infty)$ and $x_2 \in [-1,\infty)$ and $f(x_1) = f(x_2)$. [2]
Then $\sqrt{x_1+1} = \sqrt{x_2+1}$, and squaring both sides gives $x_1+1 = x_2+1$, so $x_1 = x_2$.
This proves that f is one-to-one. [2]

Now suppose $y \in [0,\infty)$. Let $x = y^2-1$. [2] Since $y^2 \geq 0$, then $y^2-1 \geq -1$, so $x \geq -1$, so $x \in [-1,\infty)$. Also $f(x) = \sqrt{(y^2-1)+1} = \sqrt{y^2} = y$. [2] So $\exists x \in [-1,\infty)$ $f(x) = y$. This proves f is onto.

b) Find f^{-1}.

[4] Setting $y = f(x)$ and solving for x gives: $y = \sqrt{x+1}$, $y^2 = x+1$, $y^2-1 = x$. So $f^{-1}(y) = y^2-1$, or $f^{-1}(x) = x^2-1$ (for all $x \in [0,\infty)$).

c) Find a formula for $f \circ f^{-1}$.

[4] $f \circ f^{-1}(x) = f(f^{-1}(x)) = f(x^2-1) = \sqrt{(x^2-1)+1} = \sqrt{x^2} = x$ (for all $x \in [0,\infty)$).

d) If $g(x) = \dfrac{x}{x+1}$, find a formula for $g \circ f$.

[4] $g \circ f(x) = g(f(x)) = g(\sqrt{x+1}) = \dfrac{\sqrt{x+1}}{\sqrt{x+1}+1}$ (for all $x \in [-1,\infty)$).

2. *(8 points)* Find the first five terms (a_0 through a_4) of the sequence defined by the the initial conditions $a_0 = 2$ and $a_1 = 3$, and the recurrence relation $a_n = a_{n-1} + na_{n-2}$, for $n \geq 2$.

$a_0 = 2$
$a_1 = 3$ 2
$a_2 = a_1 + 2 \cdot a_0 = 3 + 2 \cdot 2 = 7$ 2
$a_3 = a_2 + 3 \cdot a_1 = 7 + 3 \cdot 3 = 16$ 2
$a_4 = a_3 + 4 \cdot a_2 = 16 + 4 \cdot 7 = 44$ 2

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3. *(24 points)*
a) Fill in the table for multiplication modulo 5. Show your work.

[12]

× mod 5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

4
4
4

(mod 5)

$2\cdot 2 = 4$

$3\cdot 2 = 2\cdot 3 = 6 \equiv 1$

$4\cdot 2 = 2\cdot 4 = 8 \equiv 3$

$3\cdot 3 = 9 \equiv 4$

$4\cdot 3 = 3\cdot 4 = 12 \equiv 2$

$4\cdot 4 = 16 \equiv 1$

[12] b) Find the value of 3^{403} **mod** 5. Explain how you found your answer.

1 $3^0 \equiv 1 \pmod 5$

1 $3^1 \equiv 3 \pmod 5$

1 $3^2 \equiv 9 \equiv 4 \pmod 5$

1 $3^3 \equiv 12 \equiv 2 \pmod 5$

1 $3^4 \equiv 6 \equiv 1 \pmod 5$

After this, for 3^n for $n \geq 5$, The sequence repeats itself every time n increases by 4. So for $k = 0, 1, 2, \ldots$

4

$3^{4k+0} \equiv 1 \pmod 5$

$3^{4k+1} \equiv 3 \pmod 5$

$3^{4k+2} \equiv 4 \pmod 5$

$3^{4k+3} \equiv 2 \pmod 5$.

Since $403 = 4\cdot k + 3$ for $k = 100$, Then

3

$3^{403} \equiv 2 \pmod 5$.

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4. *(8 points)* Find the decimal expansion of $(1010111)_2$.

$(1010111)_2 = 1\cdot 1 + 1\cdot 2 + 1\cdot 4 + 0\cdot 8 + 1\cdot 16 + 0\cdot 32 + 1\cdot 64$ (4)

$= 1 + 2 + 4 + 16 + 64 = (87)_{10}$.

(4)

5. *(15 points)* Convert the following decimal numbers to other bases, using the algorithm given in class. (If you don't remember the algorithm you can get partial credit for finding the conversion by some other method.)

a) Find the base 2 expansion of $(397)_{10}$.

[8] $397 \equiv 1 \pmod 2$. Dividing by 2 and finding the remainder at each step, we get

(2) divide by 2

(2) record remainders

397	
198	R1
99	R0
49	R1
24	R1
12	R0
6	R0
3	R0
1	R1
0	R1

(2) Now reading the column of remainders from the bottom up (2) gives the base 2 expansion of 397:

$(397)_{10} = (110001101)_2$

b) Find the base 5 expansion of $(915)_{10}$.

[7] Following the above procedure but now dividing by 5 at each step gives: (2)

915	
183	R0
36	R3
7	R1
1	R2
0	R1

(3)

(2)

So $(915)_{10} = (12130)_5$

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6. *(13 points)* Find the product of $(101101)_2$ and $(110010)_2$ by performing the multiplication algorithm in base 2 (do not convert to decimals).

```
             101101
      ×      110010
      ---------------
             1011010
            0000000
           000000
          000000
         101101
        101101
      ---------------
      100011001010
```

Carried numbers are circled

5

5 add
3 carry

So $\boxed{(101101)_2 \times (110010)_2 = (100011001010)_2.}$

[check: $(101101)_2 = (45)_{10}$ and $(110010)_2 = (50)_{10}$, so their product is $(2250)_{10}$, which equals $(100011001010)_2$]

7. *(12 points)* Prove that if the decimal expansion of n is $n = a_2 \cdot 100 + a_1 \cdot 10 + a_0$, and $a_2 - a_1 + a_0 = 0$, then n is divisible by 11. *Hint:* $10 \equiv -1 \pmod{11}$.

Since $10 \equiv -1 \pmod{11}$, and $100 = 9 \cdot 11 + 1$, so $100 \equiv 1 \pmod{11}$, 3

Then $n \equiv [a_2 \cdot 1 + a_1 \cdot (-1) + a_0] \pmod{11}$

So $n \equiv (a_2 - a_1 + a_0) \pmod{11}$. 3

Since $a_2 - a_1 + a_0 = 0$,

Then $n \equiv 0 \pmod{11}$, 3

So n is divisible by 11. 3