Quiz 4

1. Determine (with explanation) whether the series is absolutely convergent, conditionally convergent, or divergent.

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 4} \]

Since \( \lim_{n \to \infty} \frac{n}{n^2 + 4} = \lim_{n \to \infty} \frac{n/n^2}{1 + 4/n^2} = 0 \)

and \( \frac{n}{n^2 + 4} \) is decreasing (you can check this by verifying \( \frac{d}{dx} \left( \frac{x}{x^2 + 4} \right) < 0 \)

for \( x > 2 \))

then the series converges by the Alt. Series Test.

For \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 4} \), we compare to \( \sum_{n=1}^{\infty} \frac{n}{n^2} \).

Since \( \lim_{n \to \infty} \frac{n}{n^2 + 4} = \lim_{n \to \infty} \frac{n/n^2}{1 + 4/n^2} = 0 \)

and \( \sum_{n=1}^{\infty} \frac{n}{n^2} \) diverges, then \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 4} \) diverges by the Limit Comparison Test. So \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 4} \) converges conditionally.

2. Find the radius of convergence and interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n} \).

Let \( a_n = \frac{n^2 x^n}{2^n} \). Then \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)^2}{2^{n+1}} x^{n+1}}{\frac{n^2 x^n}{2^n}} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{1}{x} = \frac{1}{2} \cdot \frac{1}{x} \)

Then \( \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{2} \cdot \frac{1}{1} \)

So by the Ratio Test, the series converges if \( \frac{1}{2} < 1 \), or if \( |x| < 2 \), and diverges if \( \frac{1}{2} > 1 \), or if \( |x| < 2 \).

The radius of convergence is therefore 2.

When \( x = 2 \), then the series is \( \sum \frac{n^2 x^n}{2^n} = \sum n^2 \), which diverges by the Divergence Test.

When \( x = -2 \), then the series is \( \sum \frac{(-2)^n}{2^n} = \sum (-1)^n \frac{n^2}{2^n} \), which diverges by the Divergence Test. The interval of convergence is \((-2, 2)\).