1. (12 points) Suppose \( x = t + \sin t, \ y = 1 - \cos t \). Find
   
   a) \( \frac{dy}{dx} = \frac{\left( \frac{dy}{dt} \right)^2}{\left( \frac{dx}{dt} \right)} = \frac{\sin t}{1 + \cos t} \)
   
   b) \( \frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left( \frac{\sin t}{1 + \cos t} \right)}{1 + \cos t} = \frac{\left( 1 + \cos t \right) \cos t - \sin t \sin t}{(1 + \cos t)^2} \)
   
   \[ \frac{\cos t + \cos^2 t + \sin^2 t}{(1 + \cos t)^2} = \frac{\cos t + 1}{(1 + \cos t)^3} \]

2. (18 points) The diagram shows the curve given by \( x = t^3 + t, \ y = t - t^2 \). The shaded region lies between the curve and the x-axis. Find its area.

   \[
y = 0 \quad \text{when} \quad t - t^2 = 0 \]

   \[ t(1 - t) = 0 \quad \text{or} \quad t = 0 \quad \text{or} \quad t = 1. \]

   When \( t = 0 \), \( x = 0 \); and when \( t = 1 \), \( x = 2 \).

   Area = \( \int_{x=0}^{x=2} y \, dx \) = \( \int_{t=0}^{t=1} (t-t^2)(3t^2+1) \, dt \)

   \( \) \( \) \( \) \( \) \( \)

   \( = \int_{0}^{1} \left( 3t^3 - 3t^4 + t - t^2 \right) \, dt \)

   \( = \left[ \frac{3t^4}{4} - \frac{3t^5}{5} + \frac{t^2}{2} - \frac{t^3}{3} \right]_0 \)

   \( = \left( \frac{3}{4} - \frac{3}{5} + \frac{1}{2} - \frac{1}{3} \right) - (0) = \frac{45 - 36 + 30 - 20}{60} = \frac{19}{60} \)
3. (10 points) Use an integral to find the length of the curve given by \( x = 1 + 2 \sin t \), \( y = 3 + 2 \cos t \), for \( 0 \leq t \leq \frac{\pi}{10} \).

\[
\text{length} = \int_{0}^{\pi/10} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{0}^{\pi/10} \sqrt{(2 \cos t)^2 + (-2 \sin t)^2} \, dt = 
\]

\[
= \int_{0}^{\pi/10} \sqrt{4 \cos^2 t + 4 \sin^2 t} \, dt = \int_{0}^{\pi/10} \sqrt{4} \, dt = \int_{0}^{\pi/10} 2 \, dt = 
\]

\[
\left[ 2t \right]_{t=0}^{t=\pi/10} = \frac{2\pi}{10} = \sqrt{\frac{\pi}{5}}.
\]

4. (10 points) A curve is given in polar coordinates by the equation \( r = 5 \cos \theta \).

a) Find a Cartesian equation for the curve.

\[
r = 5 \cos \theta \implies r^2 = 5r \cos \theta \implies \frac{x^2 + y^2}{r} = 5 \implies \frac{x^2}{5} + \frac{y^2}{5} = \frac{x}{5}.
\]

b) Complete the square in the Cartesian equation to put it in the form of the equation of a circle. Give the center and radius of the circle, and sketch the circle.

\[
x^2 - 5x + y^2 = 0
\]

\[
\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}
\]

This is a circle with radius \( \frac{5}{2} \) and center \( \left(\frac{5}{2}, 0\right) \).

5. (14 points) Find the slope of the tangent line to the polar curve \( r = 2 - 3 \sin \theta \) at the point where \( \theta = 0 \) (see diagram).

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2 \cos \theta - 6 \sin \theta \cos \theta}{-2 \sin \theta - (3 \cos^2 \theta + (\sin \theta)(-\sin \theta))}
\]

When \( \theta = 0 \),

\[
\frac{dy}{dx} = \frac{2 \cos 0 - 6 \sin 0 \cos 0}{-2 \sin 0 - 3 \cos^2 0 + 3 \sin^2 0} = \frac{2 - 0}{0 - 3 + 0} = \frac{-2}{3}.
\]
6. (18 points) Find the area of one loop of the curve \( r = \sin 6\theta \) (see diagram).

The loop begins at the origin at \( \theta = 0 \). The tip of the loop is at \( \theta = \frac{\pi}{12} \) \((15^{\circ})\) and \( r = 1 \); and the loop returns to the origin at \( \theta = \frac{\pi}{6} \) \((30^{\circ})\).

So one loop is covered by the range \( 0 \leq \theta \leq \frac{\pi}{6} \).

The area is

\[
\frac{1}{2} \int_0^{\pi/6} r^2 \, d\theta = \frac{1}{12} \int_0^{\pi/6} \sin^2 6\theta \, d\theta = \frac{1}{12} \int_0^{\pi/6} \sin^2 u \, du = \frac{\pi}{24}.
\]

7. (18 points) Find the area inside the lemniscate \( r^2 = 6 \cos 2\theta \) and outside the circle \( r = \sqrt{3} \).

The curves \( r^2 = 6 \cos 2\theta \) and \( r = \sqrt{3} \) \((r^2 = 3)\)

intersect when \( 6 \cos 2\theta = 3 \),

or \( \cos 2\theta = \frac{1}{2} \)

\( 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots \)

\( \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots \)

The intersection in the first quadrant is at \( \theta = \frac{\pi}{6} \). The total area will be 4 times the area in the first quadrant, which is

\[
4 \left[ \int_0^{\pi/6} \frac{1}{2} (6 \cos 2\theta) \, d\theta - \int_0^{\pi/6} \frac{1}{2} \cdot 3 \, d\theta \right] = \frac{\pi}{6} \left[ \frac{12 \sin 2\theta}{2} \right]_0^{\pi/6} - \left[ 6\theta \right]_0^{\pi/6}
\]

\[
= \left[ \frac{12 \sin 2\theta}{2} \right]_0^{\pi/6} - \left[ 6\theta \right]_0^{\pi/6}
\]

\[
= (12 \sin \frac{\pi}{3} - 0) - (6 \cdot \frac{\pi}{6} - 0)
\]

\[
= 3\sqrt{3} - \pi
\]