Quiz 5

Name: key

[5] **1.** Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-5)^n x^n}{\sqrt[3]{n}}$.

Here $a_n = \frac{(-5)^n x^n}{n^{1/3}}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-5)^{n+1} x^{n+1}}{(n+1)^{1/3}} \cdot \frac{n^{1/3}}{(-5)^n x^n} \right| =$

$$= \lim_{n\to\infty} 5|x| \left(\frac{n}{n+1}\right)^{\frac{1}{3}} = 5|x| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{3}} = 5|x| \cdot \left(\frac{1}{1+0}\right)^{\frac{1}{3}} = 5|x|.$$

By the Ratio Test, the series converges if $5|x| < 1$, or if $|x| < \frac{1}{5}$; and diverges if $|x| > \frac{1}{5}$. If $|x| = \frac{1}{5}$, there are two possibilities: either $x = 1/5$ or $x = -1/5$.

If $x = \frac{1}{5}$, the series is $\sum_{n=1}^{\infty} \frac{(-5)^n (\frac{1}{5})^n}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$. Since $\left(\frac{1}{n^{1/3}}\right)$ is a decreasing sequence which converges to zero, this series converges by the alternating series test.

If $x = -1/5$, the series is $\sum_{n=1}^{\infty} \frac{(-5)^n (\frac{-1}{5})^n}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$. This series diverges (we saw in class that any series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the integral test if $p \le 1$).

So the series converges for x in the interval $-\frac{1}{5} < x \le \frac{1}{5}$. The radius of convergence is $\frac{1}{5}$.

[5] **2.** Find a power series representation for the function $f(x) = \frac{x^2}{9 + x^3}$.

$$\frac{x^2}{9+x^3} = \frac{x^2}{9} \cdot \frac{1}{\left(1+\frac{x^3}{9}\right)} = \frac{x^2}{9}\left[1 - \left(\frac{x^3}{9}\right) + \left(\frac{x^3}{9}\right)^2 - \left(\frac{x^3}{9}\right)^3 + \left(\frac{x^3}{9}\right)^4 \ldots\right]$$

$$= \frac{x^2}{9} - \frac{x^5}{9^2} + \frac{x^8}{9^3} - \frac{x^{11}}{9^4} + \frac{x^{14}}{9^5} - \ldots$$

(the pattern is clear from this, but you can write the series as $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{3n-1}}{9^n}$ if you like.)