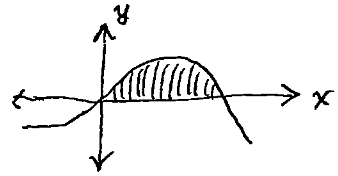
FINAL EXAM
Math 2433
5-12-11

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

~~*NOTE: In questions that ask for an infinite series, if you don't feel like giving the series in summation notation, then giving the first three nonzero terms is enough.*~~

1. *(20 points)* A curve is given parametrically by the equations $x = t + t^3$, $y = t - t^3$. (See diagram.)



a. Find the slope of the tangent line to the curve at the point where $t = 1/2$.

[8] The slope is $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1-3t^2}{1+3t^2} = \frac{1-3/4}{1+3/4} = \frac{1/4}{7/4} = \boxed{\frac{1}{7}}$ (2) (2) (2) (2)

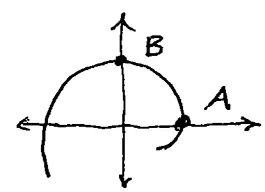
b. Find the area of the shaded region in the diagram.

[12] The curve crosses the x-axis where $y=0$, or where $t-t^3=0$, or where $t(1-t^2)=0$, or at $t=0$, $t=-1$, $t=+1$. The corresponding values of x are $x=-2$, $x=0$, $x=2$. The shaded region is between $x=0$ and $x=2$, corresponding to $t=0$ and $t=1$. The area is (2)

$$\int_{x=0}^{x=2} y\,dx = \int_{t=0}^{t=1} (t-t^3)(1+3t^2)\,dt = \int_0^1 [t - t^3 + 3t^3 - 3t^5]\,dt = \int_0^1 [t + 2t^3 - 3t^5]\,dt$$

$$= [t^2/2 + t^4/2 - t^6/2]_0^1 = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \boxed{1/2}.$$ (2) (2) (2) (2) (2)

2. *(10 points)* Shown below is part of the polar curve $r = e^{3\theta}$. Find the length of the indicated arc AB.



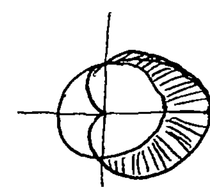
(1) A is at $\theta = 0$ and B is at $\theta = \pi/2$, so

$$\text{Length } AB = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$ (1)

$$= \int_0^{\pi/2} \sqrt{(e^{3\theta})^2 + (3e^{3\theta})^2}\,d\theta = \int_0^{\pi/2} \sqrt{e^{6\theta} + 9e^{6\theta}}\,d\theta$$ (2) (2)

$$= \int_0^{\pi/2} \sqrt{10}\, e^{3\theta}\,d\theta = \frac{\sqrt{10}}{3} e^{3\theta}\Big]_0^{\frac{\pi}{2}} = \boxed{\frac{\sqrt{10}}{3}\left[e^{3\pi/2} - 1\right]}$$ (2) (2)

3. *(20 points)* The shaded region in the diagram below lies between the curves $r = 1$ and $r = 1 + \cos\theta$. Find the area of the region.



The curves intersect where $1 = 1 + \cos\theta$, or $\cos\theta = 0$, or $\theta = \pi/2$ and $\theta = -\frac{\pi}{2}$. By symmetry the area is twice that between $\theta = 0$ and $\theta = \pi/2$. So if $r_1 = 1 + \cos\theta$ and $r_2 = 1$, then (2)

$$\text{area} = 2\left[\frac{1}{2}\int_0^{\pi/2} (r_1^2 - r_2^2)\,d\theta\right]$$ (2) (2) (2) (2)

$$= \int_0^{\frac{\pi}{2}} [(1+\cos\theta)^2 - 1^2]\,d\theta = \int_0^{\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\,d\theta$$ (2) (2)

$$= 2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta + \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 2[\sin\theta]_0^{\frac{\pi}{2}} + \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} = 2\cdot 1 + \left[\frac{\pi}{4} + 0\right]$$ (2) (2) (2)

$$= \boxed{2 + \pi/4}$$

 Name ______________________

4. *(10 points)* Use the integral test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{1}{u}\,du$$

(2) (2)

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \Big[\ln u\Big]_{u=\ln 2}^{u=\ln b} = \lim_{b\to\infty}\Big[\ln(\ln b) - \ln(\ln 2)\Big] = \infty$$, so the series diverges.

(2) (2) (2)

5. *(15 points)* Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

a. Explain how you know that the series converges.

[8] Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n}}$ is a decreasing, positive sequence, (4)
the series converges by the Alternating Series Test. (4)

b. If S is the sum of the series, how many terms of the series do we need to add to approximate S to within 3 decimal places? Justify your answer.

[7] For a series to which the alternating series test applies, the difference between the sum of the first n terms and (2) the sum of the series is less in absolute value than the absolute value of the $(n+1)^{st}$ term. Here, that means that the error of the approximation is less than $\frac{1}{\sqrt{n+1}}$ (2). So we need $\frac{1}{\sqrt{n+1}} < .001$ (2), or $\frac{1}{\sqrt{n+1}} < \frac{1}{1000}$
or $\sqrt{n+1} > 1000$
or $n+1 > 10^6$ (1)
So $\boxed{n > 10^6 \text{ is good enough}}$

6. *(15 points)* Determine whether the series converges. Give a reason for your answer.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$

[8] The series converges by comparison to (2) $\sum \frac{1}{n^2}$ (2), since $0 < \frac{1}{n^2+\sqrt{n}} < \frac{1}{n^2}$ (2), and $\sum \frac{1}{n^2}$ is a convergent series (2) (a p-series with $p = 2 > 1$).

b. $\sum_{n=1}^{\infty} \frac{n^3}{n^2+1}$

[7] The term of the series is $a_n = \frac{n^3}{n^2+1}$, and $\lim_{n\to\infty} a_n = \infty$ (3) (either by L'hopital's rule, or by observing that $\frac{n^3}{n^2+1} = \frac{n}{(1+\frac{1}{n^2})} \geq \frac{n}{2}$). So the series diverges (1) by the "test for divergence", which says that in order for a series to converge, (3) the limit of its terms must exist and equal 0.

7. *(15 points)* Find the first four terms of the Maclaurin series for $f(x) = \frac{1}{(1+x)^4}$. You may use the binomial theorem if you like.

(3)

$$1/(1+x)^4 = (1+x)^{-4} = \boxed{1 + (-4)x + \frac{(-4)(-5)}{2!}x^2 + \frac{(-4)(-5)(-6)}{3!}x^3 + \ldots}$$

(3) (3) (3) (3)

by the binomial theorem.

(Alternatively, it's not too hard to compute the derivatives of f: (1)

$f'(x) = -4(1+x)^{-5}$ (1), $f''(x) = 20(1+x)^{-6}$ (1), $f'''(x) = -120(1+x)^{-7}$; so

$f(0) = 1$ (1), $f'(0) = -4$ (1), $f''(0) = 20$ (1), $f'''(0) = -120$ (1), and by Taylor's theorem

$$\frac{1}{1+x} = 1 + (-4)\cdot x + \frac{20}{2!}x^2 + \frac{(-120)}{3!}x^3 + \ldots \text{ (8)})$$

8. *(15 points)* Consider the series $\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$.

a. Find the interval of convergence of the series, showing all work.

[10] Here $a_n = \frac{x^{3n}}{n!}$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (2) $= \lim_{n\to\infty} \left|\frac{x^{3(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{3n}}\right| =$

$= \lim_{n\to\infty} \left|\frac{x^{3n+3}}{x^{3n}} \cdot \frac{n!}{(n+1)!}\right|$ (2) $= \lim_{n\to\infty} \left|x^3 \cdot \frac{1}{n+1}\right| = |x|^3 \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0$. (2)

(2) Since $0<1$, the series converges. Since this is true for all real numbers x, the interval of convergence is $(-\infty, \infty)$. (2)

[5] b. What is the sum of the series?

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (2), so $\sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = e^{(x^3)}$. (3)

9. *(10 points)* Find parametric equations for the line which passes through the point $(4, 5, 6)$ and is parallel to the line $\frac{x-1}{2} = 3(y+1) = z$.

The line $\frac{x-1}{2} = 3(y+1) = z$, or $\frac{x-1}{2} = \frac{y+1}{(1/3)} = \frac{z}{1}$, is parallel to the vector $\langle 2, \frac{1}{3}, 1\rangle$ (2). So we want the line thru $(4,5,6)$ and parallel to $\langle 2, \frac{1}{3}, 1\rangle$ (2). Its parametric equations are

$$\boxed{\begin{aligned} x &= 4 + 2t \\ y &= 5 + (1/3)t \\ z &= 6 + t \end{aligned}}$$

(2) (2) (2)

10. *(15 points)* Find the cosine of the angle between the lines $x = y - 5 = z - 3$ and $\frac{x}{4} = y - 5 = \frac{z-3}{2}$.

The first line is parallel to $\vec{v}_1 = \langle 1, 1, 1\rangle$ (3) and the second line is parallel to $\vec{v}_2 = \langle 4, 1, 2\rangle$ (3). The angle between the lines is the angle between these vectors, whose cosine is (3) $\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} =$

$$= \left(1\cdot 4 + 1\cdot 1 + 1\cdot 2 \text{ (3)} \Big/ \sqrt{16+1+4} \text{ (3)} \sqrt{1+1+1}\right) = \frac{7}{\sqrt{21}\sqrt{3}} = \boxed{7/\sqrt{62}}$$

(simplification not necessary)

11. *(20 points)* Find an equation of the plane that passes through the points $P(1,-1,3)$, $Q(3,0,2)$, and $R(2,1,10)$.

3 3

$\overrightarrow{PQ} = \langle 2,1,-1\rangle$ and $\overrightarrow{PR} = \langle 1,2,7\rangle$ ~~also~~ are parallel to the plane, so $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We have 6

3

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 7 \end{vmatrix} = 9\vec{i} - 15\vec{j} + 3\vec{k},$$

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So an equation for the plane is $\boxed{9(x-1) - 15(y+1) + 3(z-3) = 0}$

(there are other forms for this equation.)

12. *(15 points)* Find the point at which the line $x = 2+t$, $y = 3-2t$, $z = 3t$ intersects the plane $6x+y-2z = 10$.

At the point of intersection we have 6

$$6(2+t) + (3-2t) - 2(3t) = 10,$$

or $12 + 6t + 3 - 2t - 6t = 10$, or $5 = 2t$, so $t = \frac{5}{2}$. 3

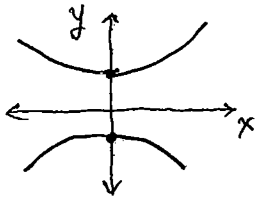
Then $x = 2 + \frac{5}{2}$, $y = 3 - 2\cdot\frac{5}{2}$, $z = 3\cdot\frac{5}{2}$, so the point is $\boxed{\left(\frac{9}{2}, -2, \frac{15}{2}\right)}$

2 2 2

13. *(20 points)* A surface is given by the equation $-4x^2 + y^2 + z^2/9 = 1$.

a. Sketch the trace of the surface in the xy-plane. What kind of conic section is this trace?

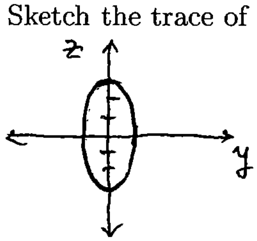
[5]



When $z=0$, the trace is $-4x^2+y^2=1$, which is a hyperbola with no x-intercepts and y-intercepts at $+1$ and -1.

b. Sketch the trace of the surface in the yz-plane. What kind of conic section is this trace?

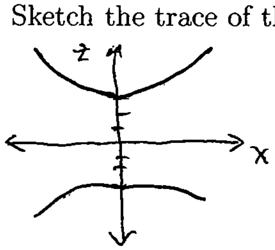
[5]



When $x=0$, the trace is $y^2 + \frac{z^2}{9} = 1$, which is an ellipse with y-intercepts at ± 1 and z-intercepts at ± 3.

c. Sketch the trace of the surface in the xz-plane. What kind of conic section is this trace?

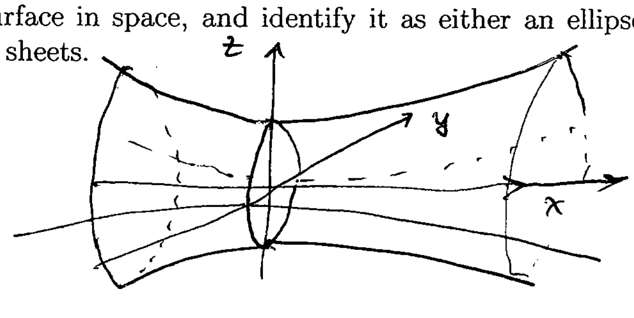
[5]



When $y=0$, the trace is $-4x^2 + \frac{z^2}{9} = 1$, which is a hyperbola with no x-intercepts and z-intercepts at $+3$ and -3.

d. Give a rough sketch of the surface in space, and identify it as either an ellipsoid, a hyperboloid of one sheet, or a hyperboloid of two sheets.

[5]



It is a hyperboloid of one sheet.