EXAM 3 — Name: key

Math 2433

4-29-11

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

NOTE: In questions that ask for an infinite series, if you don't feel like giving the series in summation notation, then giving the first three nonzero terms is enough.

1. *(18 points)* Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-1)^n}{n^3}$.

Here $a_n = \frac{3^n(x-1)^n}{n^3}$, and $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (1) $= \lim_{n\to\infty} \left| \frac{3^{n+1}(x-1)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n(x-1)^n} \right|$ (2) $=$

(2) $= \lim_{n\to\infty} 3|x-1| \left(\frac{n}{n+1}\right)^3 = 3|x-1| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^3 = 3|x-1| \cdot \left(\frac{1}{1+0}\right)^3 = 3|x-1|$. (2)

So by the Ratio Test, the series converges if $3|x-1|<1$ and diverges if $3|x-1|>1$. (2) So the series converges for $|x-1|<\frac{1}{3}$, or for $1-\frac{1}{3}<x<1+\frac{1}{3}$, or for $x\in\left(\frac{2}{3},\frac{4}{3}\right)$. The radius of convergence is $\frac{1}{3}$. (1)

If $x=\frac{2}{3}$, the series is $\sum_{n=1}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, (1)

and if $x=\frac{4}{3}$, the series is $\sum_{n=1}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$. (1) (1) Since $\sum \frac{1}{n^3}$ is a p-series with $p>1$, (1) it converges, and therefore so does $\sum \frac{(-1)^n}{n^3}$ (since $\sum \left|\frac{(-1)^n}{n^3}\right| = \sum \frac{1}{n^3}$ converges). (2) Alternatively, we could use the Alternating Series test on $\sum \frac{(-1)^n}{n^3}$. The conclusion is that the interval of convergence is (1) $\left[2/3, 4/3\right]$.

2. *(20 points)* Find power series representations for the given functions.

[8] *a)* $\dfrac{x}{1+2x^3}$

$\frac{x}{1+2x^3} = x \cdot \frac{1}{1+(2x^3)}$. Since $\frac{1}{1-r} = 1+r+r^2+r^3+\dots$, (2) taking

$r=(-2x^3)$ (2) gives $x\cdot\left[\frac{1}{1-(-2x^3)}\right] = x\left[1+(-2x^3)+(-2x^3)^2+(-2x^3)^3+\dots\right]$ (2)

$= x\left[1-2x^3+4x^6-8x^9+\dots\right]$ (2) $= \boxed{x-2x^4+4x^7-8x^{10}+\dots}$ (2)

$= \sum_{n=0}^{\infty} (-2)^n x^{3n+1}$

2. *(continued)*

b) $\dfrac{1}{\sqrt{1-x^2}}$ (Hint: use the binomial series.)

[7]

The binomial series is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \quad 2$$

$$\frac{\frac{1}{2}\cdot\frac{3}{2}}{2} = \frac{3}{8} \qquad \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{6} = \frac{15}{48} = \frac{5}{16}$$

Replacing "x" by "$-x^2$" 1 and k by $\frac{-1}{2}$ 1 2, we get

$$\frac{1}{\sqrt{1-x^2}} = [1+(-x^2)]^{-\frac{1}{2}} = 1 + \left(\tfrac{-1}{2}\right)(-x^2) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}(-x^2)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(-x^2)^3 + \dots \quad 1$$

$$= \boxed{1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots}$$

[5] *c)* $\arcsin x$ (Hint: the derivative of $\arcsin x$ is $\dfrac{1}{\sqrt{1-x^2}}$.)

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \left(1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)dx \quad 2$$

$$\Rightarrow \arcsin x = C + \cancel{x} + \frac{x^3}{6} + \frac{3}{8}\cdot\frac{x^5}{5} + \frac{5}{16}\frac{x^7}{7} + \dots \quad 2$$

Setting $x=0$, we see that $0 = C + 0 + 0 + \dots$, so $C=0$, and

$$\boxed{\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{42} + \dots} \quad 1$$

3. *(12 points)* Evaluate the definite integral $\displaystyle\int_0^1 \frac{e^x - 1 - x}{x^2}\,dx$ as an infinite series.

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The Maclaurin series for e^x is $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots$,

so

$$\frac{e^x - 1 - x}{x^2} = \frac{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots \quad 2 \quad 2$$

and

$$\int_0^1 \left(\frac{e^x - 1 - x}{x^2}\right)dx = \left[\frac{x}{2!} + \frac{x^2}{2\cdot 3!} + \frac{x^3}{3\cdot 4!} + \frac{x^4}{4\cdot 5!} + \dots\right]_0^1 = \boxed{\frac{1}{2!} + \frac{1}{2\cdot 3!} + \frac{1}{3\cdot 4!} + \frac{1}{4\cdot 5!} + \dots} \quad 2 \quad 2$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-1)n!}$$

4. *(15 points)* Find the Maclaurin series for $f(x) = \dfrac{e^x + e^{-x}}{2}$ using the definition of Maclaurin series. (Remember to give at least the first three *nonzero* terms.)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \dfrac{f^{(n)}(0)}{n!}$ 3
0	$(e^x+e^{-x})/2$ 1	$\frac{1+1}{2} = 1$ 1	1
1	$(e^x-e^{-x})/2$ 1	$\frac{1-1}{2} = 0$ 1	0
2	$(e^x+e^{-x})/2$ 1	1 1	$1/2!$
3	$(e^x-e^{-x})/2$ 1	0 1	0
4	$(e^x+e^{-x})/2$ 1	1 1	$1/4!$
5	$(e^x-e^{-x})/2$	0	0
6	$(e^x+e^{-x})/2$	1	$1/6!$

The Maclaurin series is

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$$\frac{e^x+e^{-x}}{2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= \boxed{1 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 + \frac{x^5}{5!} + \cdots}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

5. *(15 points)* Find the Taylor series for $f(x) = 1/x^2$ centered at $a = 2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$ 3	$c_n = \dfrac{f^{(n)}(2)}{n!}$ 3
0	$1/x^2 = x^{-2}$ 1	$\frac{1}{4}$ 1	$\frac{1}{4}$
1	$-2x^{-3}$ 1	$-2\cdot\frac{1}{8}$ 1	$-2\cdot\frac{1}{8}$
2	$6x^{-4}$ 1	$6\cdot\frac{1}{16}$ 1	$3\cdot\frac{1}{16}$
3	$-24x^{-5}$	$-24\cdot\frac{1}{32}$	$-4\cdot\frac{1}{32}$
4	$5!\,x^{-6}$	$5!\cdot\frac{1}{2^6}$	$5\cdot\frac{1}{2^6}$

The Taylor series is $\dfrac{1}{x^2} = c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \cdots$

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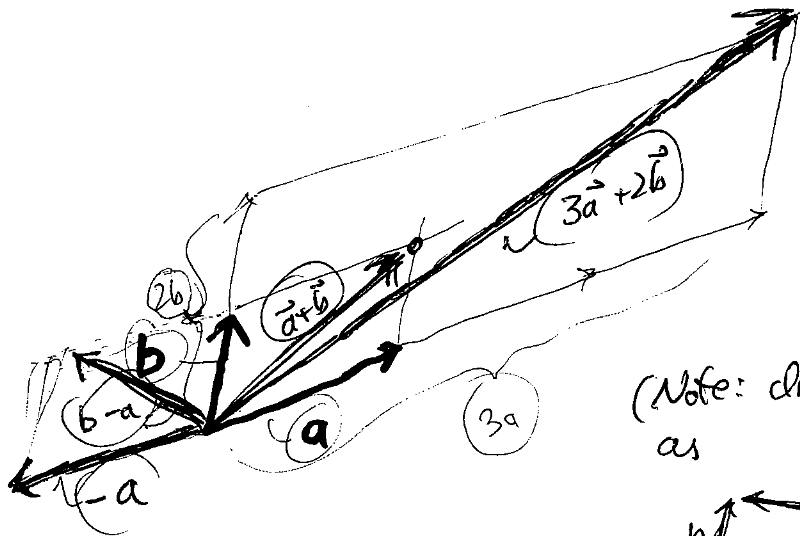
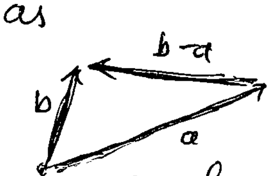
$$= \boxed{\frac{1}{4} - \frac{2(x-2)}{8} + \frac{3(x-2)^2}{16} - \frac{4(x-2)^3}{32} + \frac{5(x-2)^4}{64} - \cdots} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}} (x-2)^n$$

6. *(10 points)* The vectors **a** and **b** are shown below. In the same figure, carefully draw and label the following vectors:

(2) *a)* **a+b**
(2) *b)* **-a**
(4) *c)* **b-a**
(2) *d)* **3a+2b**



(Note: drawing $\vec{b}-\vec{a}$ as



is okay too.)

7. *(10 points)* A triangle is formed in $\mathbf{R}^3$ by the points $P(0,4,1)$, $Q(0,5,7)$, and $R(1,7,2)$.

a. Decide whether $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are orthogonal to each other. Give a reason for your answer.

[6] $\overrightarrow{PQ} = \langle 0-0, 5-4, 7-1\rangle = \langle 0,1,6\rangle$ (2)

$\overrightarrow{PR} = \langle 1-0, 7-4, 2-1\rangle = \langle 1,3,1\rangle$

so $\overrightarrow{PQ}\cdot\overrightarrow{PR} = 0\cdot 1 + 1\cdot 3 + 6\cdot 1 = 9 \neq 0$, (2)

so $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not orthogonal. (2)

[4] *b.* Is the angle between $\overrightarrow{PR}$ and $\overrightarrow{RQ}$ acute (between 0 and 90 degrees) or obtuse (between 90 and 180 degrees)? Give a reason for your answer.

$\overrightarrow{RQ} = \langle 0-1, 5-7, 7-2\rangle = \langle -1,-2,+5\rangle$,

so $\overrightarrow{PR}\cdot\overrightarrow{RQ} = 1(-1) + 3(-2) + 1\cdot 5 = -1-6+5 = -2$. (2)

Since $\overrightarrow{PR}\cdot\overrightarrow{RQ}$ is negative, then $\cos\theta = \dfrac{\overrightarrow{PR}\cdot\overrightarrow{RQ}}{|\overrightarrow{PR}||\overrightarrow{RQ}|}$ is negative,

where θ is the angle between $\overrightarrow{PR}$ and $\overrightarrow{RQ}$.

Since acute angles have positive cosine and obtuse angles have negative cosine, then θ must be obtuse. (2)