EXAM 2
Math 2433
4/1/11

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(12 points)* Find the limits of the following sequences. Briefly justify your answer.

[6] **a.** $\lim_{n\to\infty}\frac{(\sin n)^2}{n}$

③ Since $0 \le (\sin n)^2 \le 1$ for all n, then $0 \le \frac{(\sin n)^2}{n} \le \frac{1}{n}$ for all n. But $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty} 0 = 0$. So $\lim_{n\to\infty}\frac{(\sin n)^2}{n} = 0$ ② by the Squeeze Theorem ①.

[6] **b.** $\lim_{n\to\infty}\frac{(\ln n)^2}{n}$

$$\lim_{n\to\infty}\frac{(\ln n)^2}{n} \underset{\substack{\left(\frac{\infty}{\infty}\right)\\ \text{L'hopital's}}}{=} \lim_{n\to\infty}\frac{2(\ln n)\cdot\frac{1}{n}}{1}\;\text{②} = \lim_{n\to\infty}\frac{2\ln n}{n} \underset{\substack{\left(\frac{\infty}{\infty}\right)\\ \text{L'hopital's}}}{=} \lim_{n\to\infty}\frac{2\cdot\frac{1}{n}}{1}\;\text{②} = 0\;\text{②}$$

2. *(20 points)* Determine whether the series converge or diverge. Give reasons for your answers.

a. $\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}}$

[10]

We know $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges ③ (either by the integral test, or remembering that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges if $p \le 1$); and ③ $\ln n \ge 1$ for $n \ge 3$, so $\frac{\ln n}{\sqrt{n}} \ge \frac{1}{\sqrt{n}}$. Hence $\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}}$ diverges ② by the comparison test ②.

b. $\sum_{n=1}^{\infty}\frac{1}{(1.01)^n+\sin^2 n}$

[10]

For all n, $\sin^2 n \ge 0$, so $(1.01)^n + \sin^2 n \ge (1.01)^n$ ②, so $\frac{1}{(1.01)^n+\sin^2 n} \le \frac{1}{(1.01)^n} = \left(\frac{1}{1.01}\right)^n$ ②. But $1.01 > 1$, so $0 < \frac{1}{1.01} < 1$. ② Thus $\sum_{n=1}^{\infty}\left(\frac{1}{1.01}\right)^n$ is a geometric series with ratio less than 1 in absolute value, so converges. Therefore ② $\sum_{n=1}^{\infty}\frac{1}{(1.01)^n+\sin^2 n}$ converges ② by the comparison test.

3. *(20 points)*

[12] **a.** Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^4}$ converges.

NB: I will not take points off for omitting this: →

[$a_n = \frac{n}{(n^2+1)^4} \geq 0$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{n}{n^8}}{(\frac{n^2}{n^2}+\frac{1}{n^2})^4} = \lim_{n\to\infty} \frac{\frac{1}{n^7}}{(1+\frac{1}{n^2})^4} = \frac{0}{(1+0)^4} = 0$, and $f(x) = \frac{x}{(x^2+1)^4}$ is decreasing for x sufficiently large (This can be seen by computing $f'(x)$ and observing that $f'(x) < 0$ if $x > 1/\sqrt{7}$), so the integral test applies]. We have

$$\int_1^{\infty} \frac{x}{(x^2+1)^4}\,dx \text{ ② } = \frac{1}{2}\int_2^{\infty} \frac{du}{u^4} \text{ ② } = \lim_{b\to\infty} \frac{1}{2}\int_2^b u^{-4}\,du =$$

$u = x^2+1$
$du = 2x\,dx$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{u^{-3}}{-3}\right]_2^b \text{ ② } = \lim_{b\to\infty}\left[\frac{-1}{6u^3}\right]_2^b = \lim_{b\to\infty}\left[\frac{-1}{6b^3} \text{ ② } + \frac{1}{48}\right] = \frac{1}{48} < \infty \text{ ② }$$

, so the series converges. ②

[8] **b.** Show that if you add the first ten terms of the series, the result will be within 10^{-6} of the sum of the infinite series.

If S_{10} is the sum of the first ten terms, and S is the sum of the infinite series, then the difference $R_{10} = S - S_{10}$ satisfies $R_{10} \leq \int_{10}^{\infty} \frac{x}{(x^2+1)^4}\,dx$. ② From the computation above (with $u = x^2+1$) we see that

$$\int_{10}^{\infty} \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_{10^2+1}^{b} u^{-4}\,du \text{ ② } = \lim_{b\to\infty}\left[-\frac{1}{6b^3} + \frac{1}{6(10^2+1)^3}\right] = \frac{1}{6(10^2+1)^3} \text{ ② }$$

Since $\frac{1}{6(10^2+1)^3} \leq \frac{1}{(10^2+1)^3} \leq \frac{1}{(10^2)^3} = 10^{-6}$, ② then $R_n \leq 10^{-6}$.

4. *(15 points)*

[10] **a.** Find the value of the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^n}$.

We know from class that $\left(r + r^2 + r^3 + \dots = \frac{r}{1-r} \text{ ② if } |r| < 1\right)$,

So, with $r = \frac{-1}{10}$ ②, we have $\left(\frac{-1}{10} + \frac{1}{10^2} - \frac{1}{10^3} + \dots = \frac{-1/10}{1-(-1/10)}\right)$ ②.

$$\text{Thus } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^n} = \left(\frac{1}{10} - \frac{1}{10^2} + \frac{1}{10^3} - \dots\right) \text{ ② } = \frac{+1/10}{1-(-\frac{1}{10})} \text{ ② } = \frac{\frac{1}{10}}{\frac{11}{10}} = \boxed{\frac{1}{11}}$$

5] **b.** How many terms of the series in **a** would you need to add to find its sum to within 10^{-6}? Briefly justify your answer.

For an alternating series with terms a_n that decrease in absolute value, we know $|R_n| = |S - S_n| \leq |a_{n+1}|$. ② Here $|a_n| = \frac{1}{10^n}$, so $|a_{n+1}| = \frac{1}{10^{n+1}}$ ①. To get $|R_n| \leq \frac{1}{10^6}$ we need $\frac{1}{10^{n+1}} \leq \frac{1}{10^6}$ ② or $\boxed{n \geq 5.}$

Name ______________________

5. *(33 points)* Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. Give reasons for your answers.

[11] **a.** $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(1.02)^n}$ Let $a_n = \frac{(-1)^n n}{(1.02)^n}$. Then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ ② $= \lim_{n\to\infty} \frac{\left(\frac{n+1}{(1.02)^{n+1}}\right)}{\left(\frac{n}{(1.02)^n}\right)}$ ② $=$

② $= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \frac{(1.02)^n}{(1.02)^{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \left(\frac{1}{1.02}\right) = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right) \cdot \frac{1}{1.02}$

$= \frac{1}{1.02} < 1$. ② So the series converges absolutely ③ by the ratio test.

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[11] **b.** $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(\ln n)^2}$ Let $a_n = \frac{(-1)^n n}{(\ln n)^2}$; then $|a_n| = \frac{n}{(\ln n)^2}$, and

$\lim_{n\to\infty} |a_n|$ ② $= \lim_{n\to\infty} \frac{n}{(\ln n)^2} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{1}{2(\ln n)\cdot\frac{1}{n}}$ ② $= \lim_{n\to\infty} \frac{n}{2\ln n} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{1}{\left(2\cdot\frac{1}{n}\right)}$ ②

$= \lim_{n\to\infty} \frac{n}{2} = \infty$. ② Since $\lim_{n\to\infty} |a_n| = \infty$, then a_n cannot converge to zero. So the series diverges by the "test for divergence". ③

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[11] **c.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n - \sqrt{n}}$. Let $b_n = \frac{1}{2n - \sqrt{n}}$, so the series is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$.

Then $b_n \geq 0$ for all n (since $2n \geq \sqrt{n}$, or $2 \geq \sqrt{n}$); and $\lim_{n\to\infty} b_n = 0$, ②

and $b_{n+1} \leq b_n$ for all n (this can be verified by checking that $f(x) = 2x - \sqrt{x}$ has a positive derivative for $x \geq 1$, so $f(x)$ is increasing, so $1/f(x)$ is decreasing).

I will not take points off for omitting this

So $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test. ②

On the other hand, $b_n \geq$ ② $\frac{1}{2n}$, and $\sum_{n=1}^{\infty} \frac{1}{2n}$ ② diverges, so $\sum |a_n| = \sum b_n$ diverges by the comparison test. ②

So $\sum_{n=1}^{\infty} a_n$ converges conditionally. ①