EXAM 1
Math 2433
2-25-11

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(24 points)* A curve is given parametrically by the equations $x = t + e^t$, $y = t^2 + e^t$.

[8] **a.** Find dy/dx.

$$\frac{dy}{dt} = 2t + e^t \text{ and } \frac{dx}{dt} = 1 + e^t, \text{ so } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+e^t}{1+e^t}$$

2 2 2 2

[8] **b.** Find d^2y/dx^2.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2t+e^t}{1+e^t}\right)}{(1+e^t)} = \frac{\left[\frac{(1+e^t)(2+e^t)-(2t+e^t)e^t}{(1+e^t)^2}\right]}{(1+e^t)}$$

2 2 2 2

[4] **c.** What is the slope of the curve at the point where $t = 0$? Explain your answer.

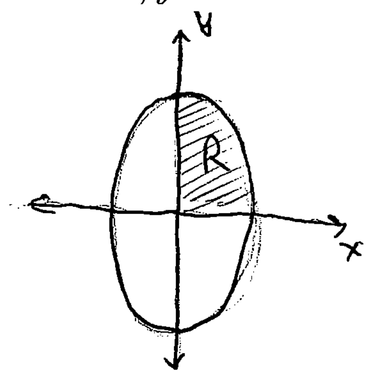
$$2 \quad \left.\frac{dy}{dx}\right|_{x=0} = \frac{2\cdot 0 + e^0}{1+e^0} = \frac{0+1}{1+1} = \frac{1}{2}. \quad 2$$

[4] **d.** Is the curve concave up or down at the point where $t = 0$? Explain your answer. 2

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{[(1+e^0)(2+e^0)-(0+e^0)e^0]}{(1+e^0)^3} = \frac{(1+1)(2+1)-(0+1)\cdot 1}{(1+1)^3} = \frac{5}{8}.$$

2 Since $\left.\frac{d^2y}{dx^2}\right|_{t=0}$ is positive, the curve is concave up.

2. *(20 points)* The curve in the diagram is given parametrically by the equations $x = 3\cos t$, $y = 5\sin t$. Find the area of the region R enclosed by the curve in the first quadrant.



$$A = \int_{x=0}^{x=3} y\,dx = \int_{t=\frac{\pi}{2}}^{t=0} (5\sin t)(-3\sin t\,dt)$$

2 2 2 2 2 2

$$= \int_{\frac{\pi}{2}}^{0} (-15)\sin^2 t\,dt = 15\int_0^{\pi/2} \sin^2 t\,dt$$

2

$$= 15\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2t)\,dt = \frac{15}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} = \frac{15}{2}\left[\frac{\pi}{2} - 0\right] = \boxed{\frac{15\pi}{4}}$$

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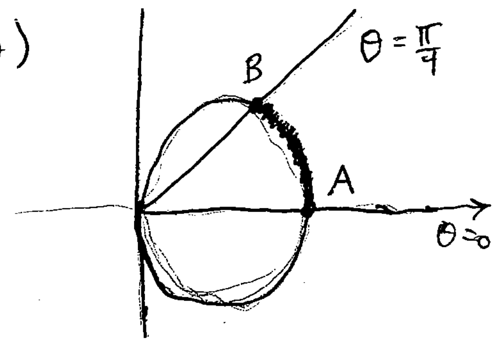
3. *(12 points)* Find the length of the portion of the polar curve $r = 5\cos\theta$ that lies between the points A and B (see diagram).



$$L = \int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad (2)\ (2)$$

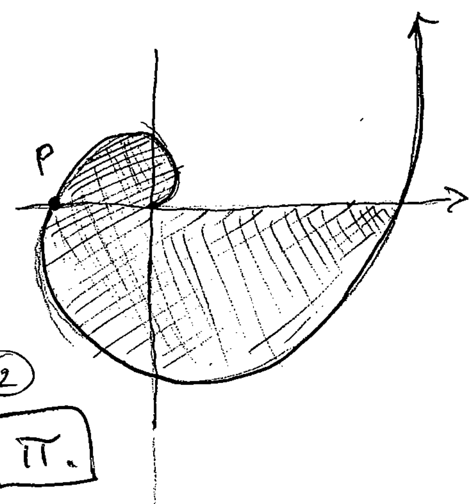
$$\left(\frac{dr}{d\theta} = -5\sin\theta\right) \quad (2)$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{25\cos^2\theta + 25\sin^2\theta}\, d\theta \quad (2)$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{25}\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = \int_0^{\pi/4} 5\, d\theta \quad (2)$$

$$= \left[5\theta\right]_0^{\pi/4} = \boxed{\frac{5\pi}{4}}. \quad (2)$$

4. *(24 points)* Shown below is part of the polar curve $r = \theta$.



a. Find the slope of the tangent to the curve at the point P (where $\theta = \pi$.

[12]

$y = r\sin\theta = \theta\sin\theta$ (2)

$x = r\cos\theta = \theta\cos\theta$ (2)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{1\cdot\sin\theta + \theta\cos\theta}{1\cdot\cos\theta - \theta\sin\theta} \quad (2)\ (2)\ (2)$$

At $\theta = \pi$, $$\frac{dy}{dx} = \frac{\sin\pi + \pi\cos\pi}{\cos\pi - \pi\sin\pi} = \frac{0 + \pi(-1)}{(-1) - \pi\cdot 0} = \boxed{\pi.} \quad (2)$$

b. Find the area of the shaded region.

[12]

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta \quad (2)\ (2)\ (2)\ (2)$$

$$= \left[\frac{\theta^3}{6}\right]_0^{2\pi} = \frac{8\pi^3}{6} = \boxed{\frac{4\pi^3}{3}}$$

(2) (2)

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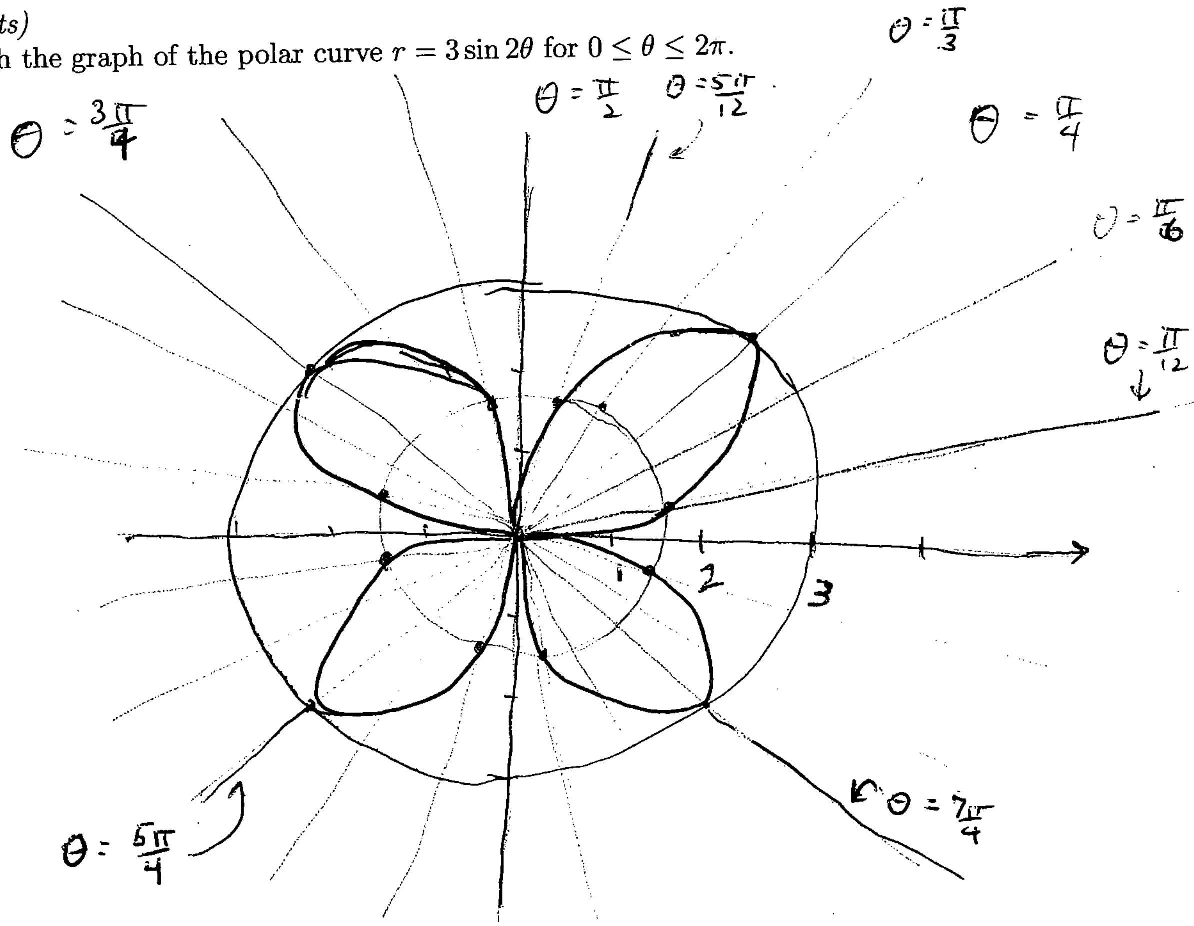
5. *(20 points)*

a. Sketch the graph of the polar curve $r = 3\sin 2\theta$ for $0 \le \theta \le 2\pi$.

[6]



b. Find the area of one leaf of the curve.

[14]

$$A = \frac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} r^2\, d\theta \quad (2)\ (2)\ (2)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (3\sin 2\theta)^2\, d\theta = \frac{1}{2}\cdot 9\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \quad (2)$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[1 - \cos 4\theta\right] d\theta \quad (2)$$

$$= \frac{9}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{9}{4}\left[\left(\frac{\pi}{2} - \frac{\sin\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right] \quad (2)$$

$$= \boxed{\frac{9\pi}{8}} \quad (2)$$