Let $S$ be a non-empty subset of $\mathbb{R}$ which is bounded below. Prove that $\inf S = -\sup \{-x: x \in S\}$.

**Proof:** By the Completeness Axiom, all non-empty subsets of $\mathbb{R}$ have infimums. Let $w = \inf S$. We want to prove that $-w = \sup \{-x: x \in S\}$.

(i) Let $x \in T$ (i.e., the following statements are true for all $x \in T$)

1. Then $\exists a \in S$, $x = -a$ \text{ [def of $T$]}
2. $-x = a$ \text{ [3]}
3. $-x \in S$ \text{ [2], [3]}
4. $-x \geq w$ \text{ [def of infimum]}
5. $x \leq -w$ \text{ [5]}
6. $-w$ is an upper bound of $T$ \text{ [1] $\Rightarrow$ [6]}

(ii) Let $\nu$ be an upper bound of $T$ (the following statements are true for all upper bounds $\nu$ of $T$)

1. Let $a \in S$ (the following statements are true for all $a \in S$)
2. $\exists a \in S$ \text{ [def of $T$]}
3. $-a \in T$ \text{ [def of $T$]}
4. $-a \leq \nu$ \text{ [1]}
5. $a \geq -\nu$ \text{ [4]}
6. $-\nu$ is a lower bound of $S$ \text{ [2] $\Rightarrow$ [5]}
7. $-\nu \leq \omega$ \text{ [def of infimum]}
8. $\nu \geq -w$ \text{ [7]}
9. for every upper bound $\nu$ of $T$, $\nu \geq -w$ \text{ [1] $\Rightarrow$ [8]}.
2.4.6. Let $A$ and $B$ be bounded non-empty subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$.

Prove that $\sup (A + B) = \sup A + \sup B$ and $\inf (A + B) = \inf A + \inf B$.

**Proof**

Let $u = \sup A + \sup B$. We need to prove two things:

1. $u$ is an upper bound for $A + B$
2. For every upper bound $v$ of $A + B$, $v \leq u$.

(a) (1) Let $x \in A + B$. Then $x = a + b$ for some $a \in A, b \in B$.

(b) (2) $u = \sup A + \sup B$.

(c) (3) Since $a \in A$, then $a \leq \sup A$ [def of supremum]

(d) (4) Since $b \in B$, then $b \leq \sup B$ [def of supremum]

(e) (5) $a + b \leq \sup A + \sup B$ [add (3) and (4)]

(f) (6) $a + b \leq u$ [def of $u$]

(g) (7) $x \leq u$ [add (2)]

(h) (8) $u$ is an upper bound of $A + B$ [since (7) $\Rightarrow (2)$]

(i) (1) Let $v$ be an upper bound of $A + B$

(j) (2) Let $a \in A$

(k) (3) Let $b \in B$

(l) (4) $a + b \in A + B$ [def of $A + B$]

(m) (5) $a + b \leq v$ [add $a$ and subtract $a$ from (5)]

(n) (6) $b \leq v - a$ [subtract $a$ from (5)]

(o) (7) $v - a$ is an upper bound of $B$ [since (3) $\Rightarrow (6)$]

(p) (8) $v - a \geq \sup B$ [def of supremum]

(q) (9) $v - \sup B \geq a$ [add $a$ to both sides of (8)]

(r) (10) $v - \sup B \geq \sup A$ [since (2) $\Rightarrow (9)$]

(s) (11) $v - \sup B \geq \sup A$ [def of supremum]
(12) \( u \geq \sup A + \sup B \) \hspace{1cm} [\text{add } \sup B \text{ to (11)}]
(13) \( u \geq u \) \hspace{1cm} [\text{def } \leq ]
(14) \text{for every upper bound } u \text{ of } A + B, u \geq u \hspace{1cm} [\text{(1)} \Rightarrow \text{(13)}]

Proof that \( \inf (A + B) = \inf A + \inf B \)

We could repeat the above proof with inequalities reversed and "\( \sup \)" replaced by "\( \inf \)" throughout.

Or, we could write:

\[ -S = \{ \alpha \beta : \alpha \in A, \beta \in B \} \]

Note that \( -(A+B) = (-A) + (-B) \)

Hence if \( K \subseteq A + B \) then \( K \subseteq -\inf (-S) \). For \( S \subseteq \mathbb{R} \), define \( -S = \{ -\alpha : \alpha \in S \} \). From 2.35, we knew that if \( S \) is bounded above \( \inf \) \( S = -\sup(-S) \). Further, \( \inf (A + B) = \inf (-S) \).

So by Lemma,

\[ \inf (A + B) = \inf (-S) = -\sup(-S) = \inf (-A) + \inf (-B) \]

By 2.3.5, \( \inf (A + B) = \inf A + \inf B \).