Twists for positroid cells

Greg Muller, joint with David Speyer

July 29, 2015

Column matroids

We study $k \times n$ complex matrices, guided by the perspective:

a $k \times n$ matrix over $\mathbb{C}$ $\sim$ an ordered list of n-many vectors in $\mathbb{C}^k$

Natural question

Which subsets of the vectors form a basis?

The **column matroid** of a $k \times n$ matrix is the set of k-element subsets of $\{1, 2, ..., n\}$ which index linearly independent columns.

Example

The column matroid of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \{\{1, 3\}, \{2, 3\}\}$

Positroids

A **totally non-negative matrix** is a real-valued matrix whose minors are all non-negative.

Definition: Positroid

A **positroid** is the column matroid of a totally non-negative matrix.

Unimportant example: a non-positroid

The column matroid of $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$

is not the column matroid of any totally non-negative 2×4 matrix.

Proposition [Postnikov]

The column matroid of a matrix A is contained in a unique minimal positroid, which we call the **positroid** of A.

Positroid cells in the Grassmannian

We pass to the (k, n)-Grassmannian with the isomorphism

$$GL(k)\backslash\{k \times n \text{ matrices of rank } k\} \xrightarrow{\sim} Gr(k, n)$$

$$A \mapsto [A] := \text{span(rows of } A) \in Gr(k, n)$$

This lets us stratify $Gr(k, n)$ into strata indexed by positroids.

Definition: Positroid cell

The **positroid cell** of a positroid $\mathcal{M}$ is

$$\Pi^\circ(\mathcal{M}) := \{[A] \in Gr(k, n) \mid (\text{the positroid of } A) = \mathcal{M}\}$$

Basic example: The big positroid cell

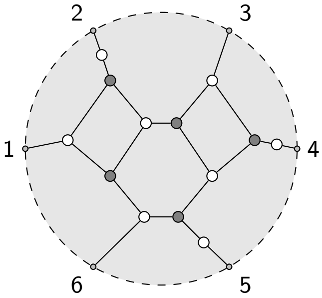
Let $\mathcal{M} = \{\text{all } k\text{-element subsets of } \{1, 2, ..., n\}\}$.

$$\Pi^\circ(\mathcal{M}) = \{[A] \in Gr(k, n) \mid \forall 1 \leq i \leq n,\ \Delta_{i,i+1,...,i+k}(A) \neq 0\}$$

2-colored graphs in the disc

Let G be a graph in the disc with...

- a bipartite 2-coloring of its internal vertices, and
- a clockwise indexing of its boundary vertices from 1 to n.

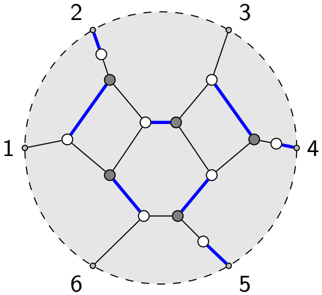


Assumptions (for simplicity)

1. Boundary vertices are next to one white vertex, and no black or boundary vertices.
2. Internal vertices have degree at least 2.
3. Components are connected to the boundary.

Matchings of G

For this talk, a **matching** of G is a subset of the edges for which every internal vertex is in exactly one edge.



Easy observation

Every matching of G contains

$k := |\text{white vertices}| - |\text{black vertices}|$

boundary vertices.

Thus, each matching determines a k-element subset of $\{1, 2, ..., n\}$.

Natural question

Given G, which k-element subsets of $\{1, 2, ..., n\}$ index the boundary of one or more matchings of G?

The positroid of G

Remarkably, the answer is a new characterization of positroids!

Theorem [essentially Postnikov]

The set of subsets of $\{1, 2, ..., n\}$ which index the boundaries of matchings of G is a positroid. Every positroid occurs this way.

A graph G is **reduced** if it has the minimal number of faces among all graphs with the same positroid as G.

..."essentially"? [Talaska, Postnikov-Speyer-Williams]

We work with **matchings in bipartite graphs** rather than the equivalent theory of **flows in perfectly oriented networks**.

Two maps, both alike in dignity

In an unpublished preprint in 2003, Postnikov associates two maps to a reduced graph with positroid $\mathcal{M}$.

- A boundary measurement map

$$\mathbb{B} : \text{an algebraic torus} \longrightarrow \Pi^\circ(\mathcal{M})$$

- A cluster: a rational map

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow \text{an algebraic torus}$$

Significance

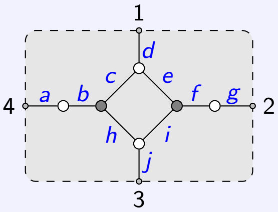
These maps have since proven fruitful in studying the combinatorics of positroids, the geometry of positroid cells, and applications to integrable systems and perturbative field theories.

Despite abundant applications, basic questions about these maps remained open for more than a decade; like, how are they related?

Partition functions

Given a k-element subset I of $\{1, 2, \ldots, n\}$, we can encode all the matchings with boundary I into a partition function Z_I.

Example: Partition functions



The $\binom{4}{2}$ partition functions are:

$$Z_{12} = bdgi \qquad Z_{13} = bdfj$$
$$Z_{14} = adfh \qquad Z_{23} = begj$$
$$Z_{24} = acgi + aegh \qquad Z_{34} = acfj$$

Easy: a partition function $Z_I = 0$ iff I is not in the positroid of G.

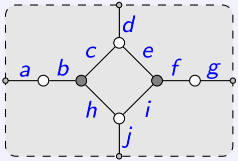
Natural question

What other relations hold among the partition functions?

Relations between partition functions

Remarkably, the partition functions satisfy the Plücker relations!

As a consequence, there is a matrix-valued function...



$$\longmapsto \begin{bmatrix} bd & \frac{bge}{f} & 0 & -ac \\ 0 & gi & fj & \frac{afh}{b} \end{bmatrix}$$

...whose Ith maximal minor equals the partition function Z_I.

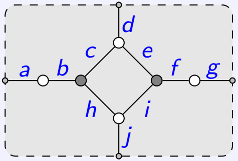
$$\begin{array}{ll} Z_{12} = bdgi & Z_{13} = bdfj \\ Z_{14} = adfh & Z_{23} = begj \\ Z_{24} = acgi + aegh & Z_{34} = acfj \end{array}$$

$$\begin{array}{ll} \Delta_{12} = bdgi & \Delta_{13} = bdfj \\ \Delta_{14} = adfh & \Delta_{23} = begj \\ \Delta_{24} = acgi + aegh & \Delta_{34} = acfj \end{array}$$

Gauge transformations

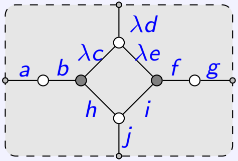
If we treat the variables as arbitrary non-zero complex numbers and compose with the projection to $Gr(k,n)$, we get a well-defined map

$$(\mathbb{C}^*)^{Edges(G)} \longrightarrow Gr(k,n)$$



$$\longmapsto \operatorname{span}\left\{\left(bd, \tfrac{bge}{f}, 0, -ac\right), \left(0, gi, fj, \tfrac{afh}{b}\right)\right\}$$

Gauge transformations: The image does not change when scaling the numbers at each edge adjacent to a fixed vertex by $\lambda \in \mathbb{C}^*$.



$$\longmapsto \operatorname{span}\left\{\left(\lambda bd, \lambda\tfrac{bge}{f}, 0, -\lambda ac\right), \left(0, gi, fj, \tfrac{afh}{b}\right)\right\}$$

The boundary measurement map

Let $(\mathbb{C}^*)^{Edges(G)}/Gauge$ be the quotient by gauge transformations. This is an algebraic torus: isomorphic to $(\mathbb{C}^*)^n$ for some n.

Theorem [Postnikov, M-Speyer]

For reduced graphs G, the map $(\mathbb{C}^*)^{Edges(G)} \to Gr(k,n)$ descends to a well-defined map of varieties

$$\mathbb{B} : (\mathbb{C}^*)^{Edges(G)}/Gauge \longrightarrow \Pi^\circ(\mathcal{M})$$

where $\mathcal{M}$ is the positroid of G.

The map $\mathbb{B}$ is called the **boundary measurement map**.

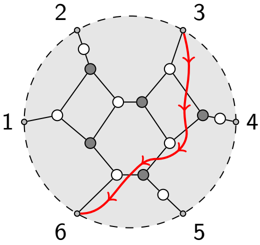
Conjecture

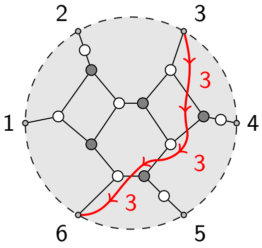
The map $\mathbb{B}$ is an open inclusion.

Strands in a reduced graph

A **strand** in reduced G is a path which...

- begins and ends at boundary vertices
- passes through the midpoints of edges
- alternates turning right around white vertices and left around black vertices



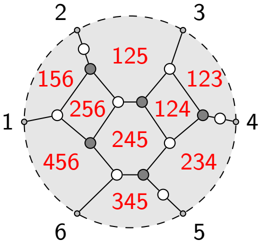


Face labels from a strand

Index a strand by its source vertex, and label each face to the left of the strand by that label.

Face labels and the cluster structure

Repeating this for each strand, each face of G gets labeled by a subset of $\{1, 2, \ldots n\}$.



Plücker coordinates of faces

Each face label has k elements, and hence defines a Plücker coordinate on $\Pi^\circ(\mathcal{M})$.

Conjecture [essentially Postnikov]

The homogeneous coordinate ring of $\Pi^\circ(\mathcal{M})$ is a cluster algebra, and the Plücker coordinates of the faces of G form a cluster.

The conjecture implies the Plückers of the faces give a rational map

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow (\mathbb{C}^*)^{Faces(G)}/Scaling$$

which is an isomorphism on its domain (the cluster torus).

Two conjectural tori associated to a reduced graph

So, a reduced graph G with positroid $\mathcal{M}$ determines two maps, and each map conjecturally defines an open algebraic torus in $\Pi^\circ(\mathcal{M})$.

- The image of the boundary measurement map

$$\mathbb{B} : (\mathbb{C}^*)^{Edges(G)} / Gauge \longrightarrow \Pi^\circ(\mathcal{M})$$

- The domain of definition of the cluster of Plücker coordinates

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow (\mathbb{C}^*)^{Faces(G)} / Scaling$$

Natural question

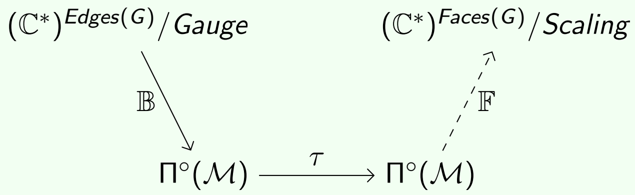
What is the relation between these two subvarieties?

In a simple world, they'd coincide and we'd have an isomorphism

$$\mathbb{F} \circ \mathbb{B} : (\mathbb{C}^*)^{Edges(G)} / Gauge \xrightarrow{\sim} (\mathbb{C}^*)^{Faces(G)} / Scaling$$

The need for a twist

In the real world, we need a **twist** automorphism τ of $\Pi^\circ(\mathcal{M})$, which will fit into a composite isomorphism



and thus take the image of $\mathbb{B}$ to the domain of $\mathbb{F}$.

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

Hence, $\tau(A)_i$ is defined by its dot product with the 'first' basis of columns of A encountered starting at column i and moving right.

Example of a twist

Example: Twisting a matrix

Consider the 3×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first column $\tau(A)_1$ of the twist is a 3-vector v, such that...

$v \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1, \quad v \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0, \quad v \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is already fixed, and $\quad v \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$

We see that $v = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. In this way, we compute the twist matrix

$$\tau(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

The twist on a positroid cell

Twisting matrices descends to a well-defined map of sets

$$Gr(k,n) \xrightarrow{\tau} Gr(k,n)$$

However, this map is not continuous; the defining equations jump when A_j deforms to a column in the span of $\{A_i, A_{i+1}, ..., A_{j-1}\}$.

Theorem [M-Speyer]

The domains of continuity of τ are precisely the positroid cells. The twist τ restricts to a regular automorphism of $\Pi^\circ(\mathcal{M})$.

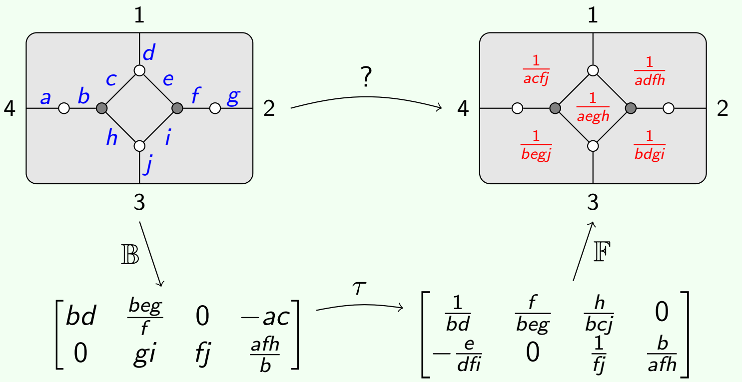
The inverse of τ is given by a virtually identical formula to τ, by reversing the order of the columns.

The induced map on tori

Let's consider our conjectural isomorphism of tori.

$$\mathbb{F} \circ \tau \circ \mathbb{B} : (\mathbb{C}^*)^{Edges(G)}/Gauge \longrightarrow (\mathbb{C}^*)^{Faces(G)}/Scaling$$

Example: the open cell of $Gr(2,4)$



Minimal matchings

It looks like the entries are reciprocals of matchings!

Lemma [M-Speyer]

Given a face F in G, there is a matching M_F such that

$$\text{the } F\text{-coordinate of } \mathbb{F} \circ \tau \circ \mathbb{B} = \frac{1}{\text{product of edges in } M_F}$$

What is the matching M_F?

The matching M_F is the **minimal** matching whose boundary is the face label of F. There is an explicit construction using strands.

The isomorphism of model tori

We can collect these coordinates into a map

$$(\mathbb{C}^*)^{Edges(G)} \longrightarrow (\mathbb{C}^*)^{Faces(G)}$$

Lemma [M-Speyer]

The above map induces an isomorphism of algebraic tori

$$\mathbb{D} : (\mathbb{C}^*)^{Edges(G)} / Gauge \longrightarrow (\mathbb{C}^*)^{Faces(G)} / Scaling$$

What is $\mathbb{D}^{-1}$?

The inverse $\mathbb{D}^{-1}$ may be induced from the map

$$(\mathbb{C}^*)^{Faces(G)} \longrightarrow (\mathbb{C}^*)^{Edges(G)}$$

such that, at each edge e

$$\text{coordinate at } e := \prod_{\text{adjacent faces } f} (\text{coordinate at } f)^{-1}$$

Putting it all together

Theorem [M-Speyer]

For each reduced graph G, there is a commutative diagram

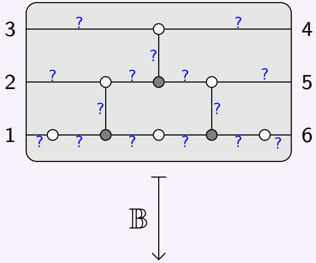
$$\begin{array}{ccc}
(\mathbb{C}^*)^{\mathit{Edges}(G)}/\mathit{Gauge} & \underset{\mathbb{D}^{-1}}{\overset{\mathbb{D}}{\rightleftarrows}} & (\mathbb{C}^*)^{\mathit{Faces}(G)}/\mathit{Scaling} \\
\mathbb{B}\downarrow & & \uparrow \mathbb{F} \\
\Pi^\circ(\mathcal{M}) & \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} & \Pi^\circ(\mathcal{M})
\end{array}$$

- The image of $\mathbb{B}$ and the domain of $\mathbb{F}$ are open algebraic tori in $\Pi^\circ(\mathcal{M})$, and τ takes one to the other.
- The (rational) inverse of $\mathbb{B}$ is $\mathbb{D}^{-1} \circ \mathbb{F} \circ \tau$.
- The (regular) inverse of $\mathbb{F}$ is $\tau \circ \mathbb{B} \circ \mathbb{D}^{-1}$.

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell



$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

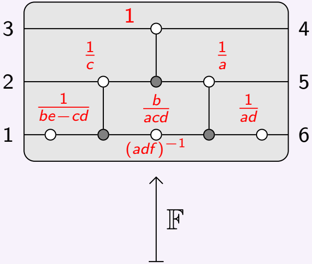
Example: The unipotent cell in $GL(3)$, as a positroid cell

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell

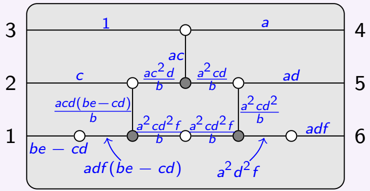


$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \overset{\mathcal{T}}{\longmapsto} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

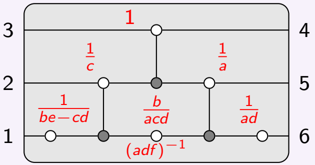
Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell



$\mathbb{D}^{-1}$ ⟵



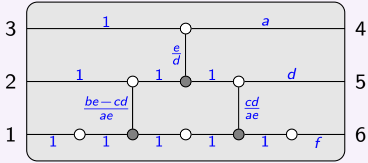
↑ $\mathbb{F}$

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \overset{\mathcal{T}}{\longmapsto} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

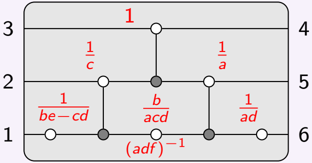
Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell



$\mathbb{D}^{-1}$ $\longleftarrow$

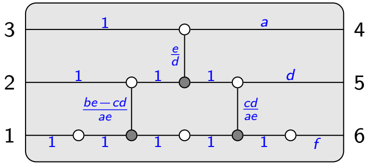


$\mathbb{B} \downarrow$ $\qquad$ $\uparrow \mathbb{F}$

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\ \mathcal{T}\ } \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Relation to the Chamber Ansatz

So, we have the following boundary measurement map.



$$\overset{\mathbb{B}}{\longmapsto} \begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

This is equivalent to a factorization into elementary matrices.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 & \frac{cd}{ae} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{e}{d} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{be-cd}{ae} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Our computation to find this factorization is identical to the *Chamber Ansatz* introduced by Berenstein-Fomin-Zelevinsky.

Connection: Monodromy coordinates

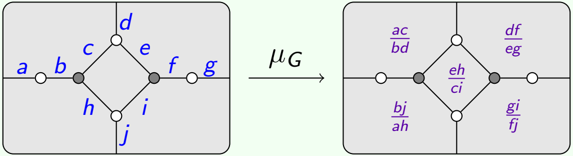
The twist also illuminates the connection between cluster coordinates and **monodromy coordinates**.

Definition: Monodromy around a face

Given a point in $(\mathbb{C}^*)^{Edges(G)}$, the **monodromy** around a face in G is the alternating product of the edge weights around that face.

Gauge transformations preserve the monodromy, and so the monodromy coordinates may be combined into a map

$$\mu_G : (\mathbb{C}^*)^{Edges(G)}/Gauge \longrightarrow (\mathbb{C}^*)^{Faces(G)}$$



The monodromy map twists to a Foch-Goncharov map

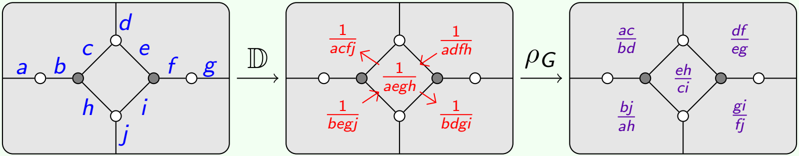
Proposition

For a face F in G, the F-coordinate of $\mu_G \circ \mathbb{D}^{-1}$ is the alternating product of adjacent faces (times a unit if F is on the boundary).

So, $\mu_G \circ \mathbb{D}^{-1}$ may be described cluster theoretically as the map

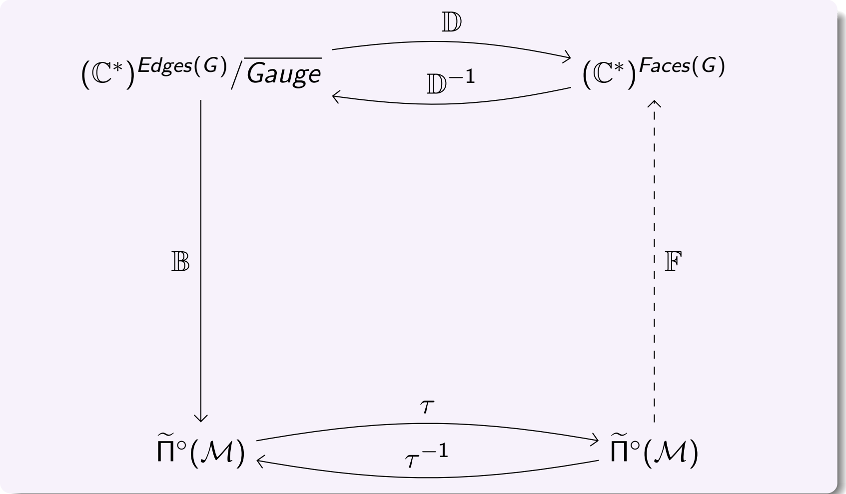
$$\rho_G : \mathcal{A}\text{-torus} \to \mathcal{X}\text{-torus}$$

defined by a square extension of the extended exchange matrix.



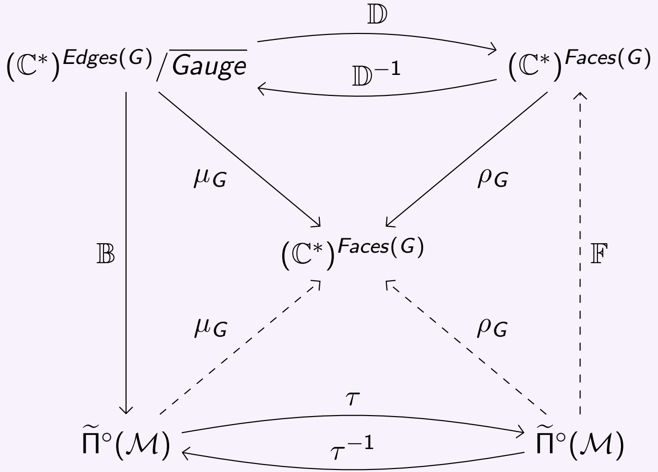
A better commutative diagram

First, we need the *affine cone* version of the original diagram.



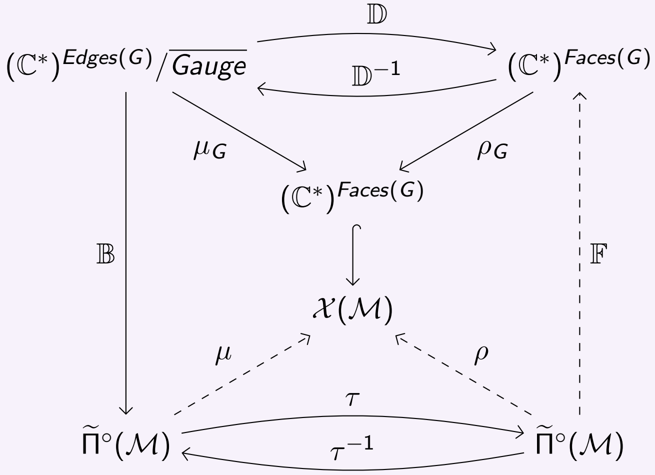
A better commutative diagram

The monodromy μ_G and Foch-Goncharov map ρ_G fit inside.



A better commutative diagram

Assuming the cluster structure, this extends to the $\mathcal{X}$-variety.



Application: Counting matchings

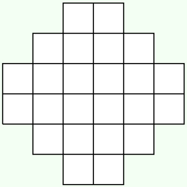
Corollary

Let G be a reduced graph with positroid $\mathcal{M}$. If A is a matrix with

- *the matroid of A is contained in $\mathcal{M}$, and*
- *for each face label I of G, the minor $\Delta_I(A) = 1$,*

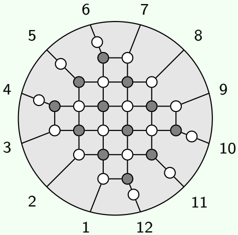
then $\Delta_J(\tau^{-1}(A))$ counts matchings with boundary J.

Example: Domino tilings of the Aztec diamond of order 3



Domino tilings of this shape...

...are the same as matchings of this graph, with boundary $\{4, 5, 6, 10, 11, 12\}$.



Application: Counting matchings

Example: (continued)

Here is an appropriate A and its inverse twist.

$$A = \begin{bmatrix} 1 & 6 & 18 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 3 & 5 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 5 & 13 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 18 & 2 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 2 & 6 & 10 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 10 & 26 & 1 & 0 & 0 \end{bmatrix}$$

$$\tau^{-1}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 10 & 6 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -6 & -18 & -26 & -10 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 5 & 13 & 18 & 6 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -3 & -5 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

We compute that $\Delta_{\{4,5,6,10,11,12\}}(\tau^{-1}(A)) = 64$. ✓

Finding A by brute force is probably not efficient, but verifying that a matrix has the necessary properties can be faster than counting.

Further directions

- Hypotheses to weaken:
  - Reduced graphs in the disc $\to$ 'reduced graphs' in surfaces.
  - Positroid cells in $Gr(k,n)$ $\to$ projected Richardson cells in partial flag varieties.
- Leclerc recently produced a cluster structure on the coordinate ring of $\widetilde{\Pi}^{\circ}(\mathcal{M})$ with categorical tools. How does this relate to given conjectural cluster structure?
- We may write any twisted Plücker coordinate as a sum over matchings of monomials in face Plückers. What is the relation to similar formulae (*snake graphs*, *Aztec everything*, etc)?
- Conjecture: The twist is the decategorification of the shift functor in an additive or Frobenius categorification.
  A cluster algebra with a Jacobi-finite potential has such a **shift** automorphism. Can this story can be extended?