Separating good cluster algebras from bad ones

Greg Muller

University of Michigan

January 25, 2015

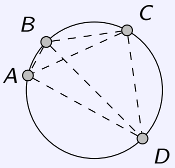
Cluster algebras are a beautiful class of commutative rings with extra combinatorial structure. However, before we see a general definition, let's motivate this structure with some examples.

Let's go back to the dawn of math: Greek geometry!

Ptolemy's theorem

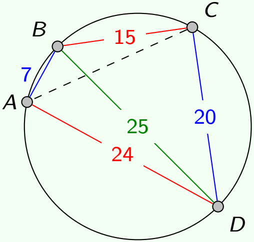
Let A, B, C, D be 4 distinct points inscribed on a circle in cyclic order. The 6 distances between these 4 points are related by

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$$



Hence, we only need to measure five of the lengths to find them all.

Example

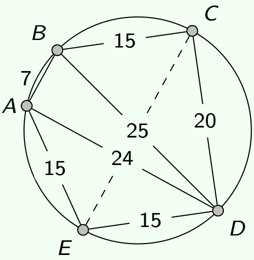


$$|AC| = \frac{|AB|\cdot|CD| + |AD|\cdot|BC|}{|BD|} = \frac{7\cdot 20 + 24\cdot 15}{25} = 20$$

Among n points on a circle, there are $\binom{n}{4}$-many Ptolemy identities. Now which measurements determine the rest?

A **triangulation**: a maximal non-crossing subset of edges

Example

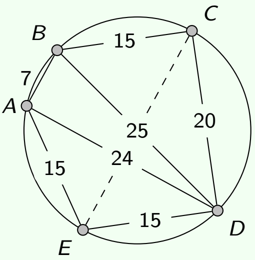


$$|CE| = ?$$

Among n points on a circle, there are $\binom{n}{4}$-many Ptolemy identities. Now which measurements determine the rest?

A **triangulation**: a maximal non-crossing subset of edges

Example

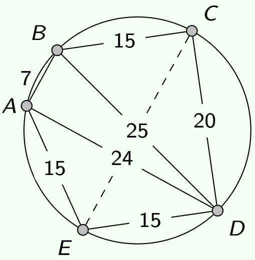


$$|CE| \quad = \quad \frac{|AC|\cdot|DE|+|AE|\cdot|CD|}{|AD|}$$

Among n points on a circle, there are $\binom{n}{4}$-many Ptolemy identities. Now which measurements determine the rest?

A **triangulation**: a maximal non-crossing subset of edges

Example

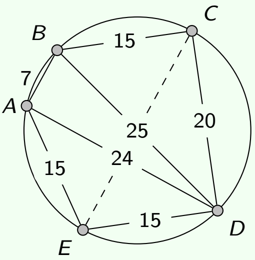


$$|CE| = \frac{\left(\frac{|AB|\cdot|CD|+|AD|\cdot|BC|}{|BD|}\right)\cdot|DE|+|AE|\cdot|CD|}{|AD|}$$

Among n points on a circle, there are $\binom{n}{4}$-many Ptolemy identities. Now which measurements determine the rest?

A **triangulation**: a maximal non-crossing subset of edges

Example

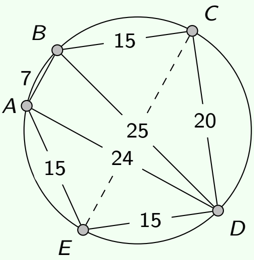


$$|CE| \quad = \quad \frac{|AB|\cdot|CD|\cdot|DE|+|AD|\cdot|BC|\cdot|DE|+|AE|\cdot|BD|\cdot|CD|}{|AD|\cdot|BD|}$$

Among n points on a circle, there are $\binom{n}{4}$-many Ptolemy identities. Now which measurements determine the rest?

A **triangulation**: a maximal non-crossing subset of edges

Example



$$|CE| = \frac{|AB|\cdot|CD|\cdot|DE|+|AD|\cdot|BC|\cdot|DE|+|AE|\cdot|BD|\cdot|CD|}{|AD|\cdot|BD|}$$

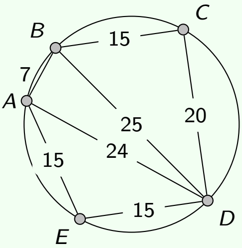
$$= \frac{7\cdot 20\cdot 15+24\cdot 15\cdot 15+15\cdot 25\cdot 20}{24\cdot 25}$$

$$= 25$$

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

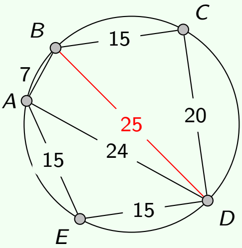


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

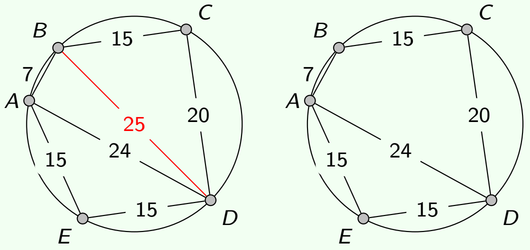


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

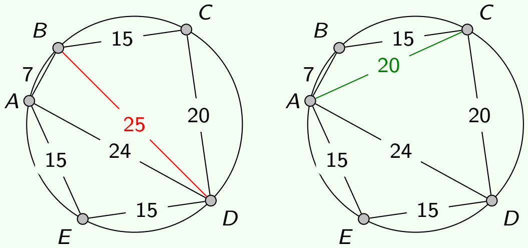


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

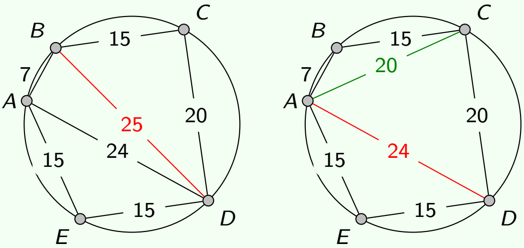


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

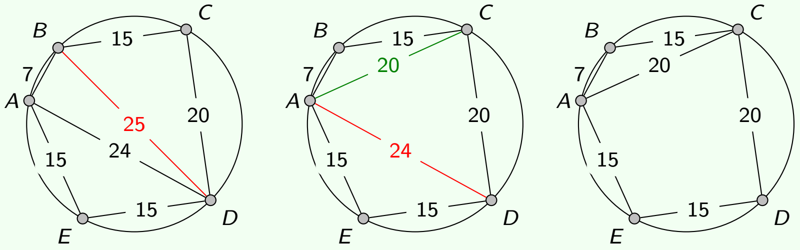


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example

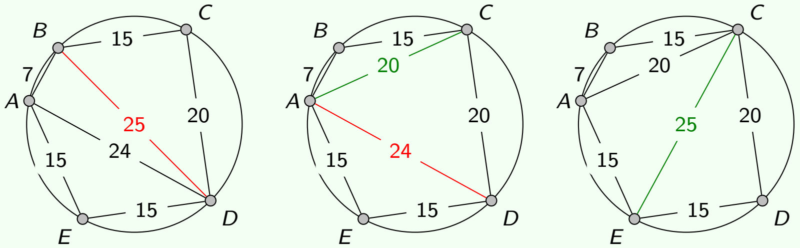


Any two triangulations are related by a sequence of adjacent ones.

Adjacent triangulations: differ in a single edge

Adjacent triangulations are related by a single Ptolemy formula.

Example



Any two triangulations are related by a sequence of adjacent ones.

Next, consider the 2×2 minors of a $2 \times n$ complex matrix A:

$$\forall i < j, \quad \Delta_{ij} := a_{1i}a_{2j} - a_{1j}a_{2i}$$

There are Plücker relations among these $\binom{n}{2}$ numbers:

$$\forall i < j < k < l, \quad \Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$$

Example

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \Delta_{13} &= \frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}}{\Delta_{24}} \\ &= \frac{4 \cdot 1 + 2 \cdot 1}{3} = 2 \end{aligned}$$

Similar questions

- Which minors of A determine the rest?
- What are the relations between these sets of minors?

However, the answers are essentially the same as the first problem, because the two sets of relations are formally equivalent!

$$\{\text{Ptolemy relations}\} \longleftrightarrow \{\text{Plücker relations}\}$$

That is, any identity or formula derived from one set of relations holds for the other set of relations.

We may study the Ptolemy/Plücker relations abstractly by studying the commutative ring they define.

The cluster algebra of the n-gon

Define a commutative ring $\mathcal{A}_n$ generated by the variables

$$\{a_{ij} \mid \forall i, j \text{ with } 1 \leq i < j \leq n\}$$

and with relations generated by

$$\{a_{ik}a_{jl} = a_{ij}a_{kl} + a_{il}a_{jk} \mid \forall i, j, k, l \text{ with } i < j < k < l\}$$

Any relation which holds in $\mathcal{A}_n$ gives a relation which holds for distances between n points on a circle and minors of a $2 \times n$ matrix.

The generators $\{a_{ij}\}$ can be identified with the edges in an n-gon.

A **triangulation**: a maximal non-crossing subset of $\{a_{ij}\}$

Does a triangulation determine the rest of $\mathcal{A}_n$ in an abstract sense?

Three equivalent questions:

- If the variables in a triangulation $T \subset \{a_{ij}\}$ take known values, can the value of any $a \in \mathcal{A}_n$ be computed?

The generators $\{a_{ij}\}$ can be identified with the edges in an n-gon.

A **triangulation**: a maximal non-crossing subset of $\{a_{ij}\}$

Does a triangulation determine the rest of $\mathcal{A}_n$ in an abstract sense?

Three equivalent questions:

- If the variables in a triangulation $T \subset \{a_{ij}\}$ take known values, can the value of any $a \in \mathcal{A}_n$ be computed?
- Can every element of $\mathcal{A}_n$ be written as a polynomial in T?

The generators $\{a_{ij}\}$ can be identified with the edges in an n-gon.

A **triangulation**: a maximal non-crossing subset of $\{a_{ij}\}$

Does a triangulation determine the rest of $\mathcal{A}_n$ in an abstract sense?

Three equivalent questions:

- If the variables in a triangulation $T \subset \{a_{ij}\}$ take known values, can the value of any $a \in \mathcal{A}_n$ be computed?
- Can every element of $\mathcal{A}_n$ be written as a polynomial in T?
- Does a triangulation $T \subset \{a_{ij}\}$ generate $\mathcal{A}_n$?

The generators $\{a_{ij}\}$ can be identified with the edges in an n-gon.

A **triangulation**: a maximal non-crossing subset of $\{a_{ij}\}$

Does a triangulation determine the rest of $\mathcal{A}_n$ in an abstract sense?

Three equivalent questions:

- If the variables in a triangulation $T \subset \{a_{ij}\}$ take known values, can the value of any $a \in \mathcal{A}_n$ be computed?
- Can every element of $\mathcal{A}_n$ be written as a polynomial in T?
- Does a triangulation $T \subset \{a_{ij}\}$ generate $\mathcal{A}_n$?

No, but yes in a weaker sense.

The trick is to weaken the notion of 'polynomial'.

$$\text{A } \textbf{Laurent polynomial} \text{ in } T := \frac{\text{a polynomial in } T}{\text{a monomial in } T}$$

Theorem (Laurent phenomenon)

Given a triangulation $T \subset \{a_{ij}\}$, every element of $\mathcal{A}_n$ can be written as a Laurent polynomial in T.

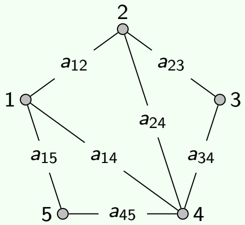
If the variables in T take known non-zero values, then the value of any $a \in \mathcal{A}_n$ can be computed by evaluating a Laurent polynomial.

Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example

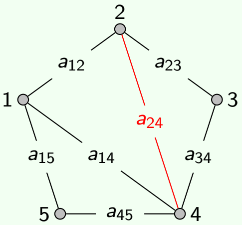


Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example

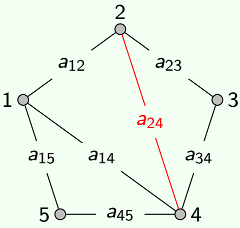


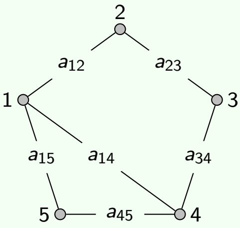
Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example



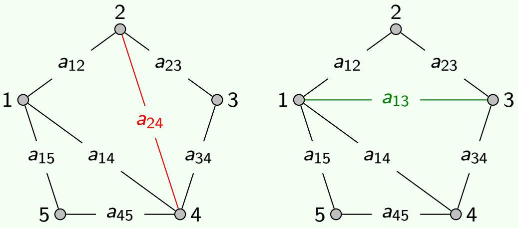


Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example



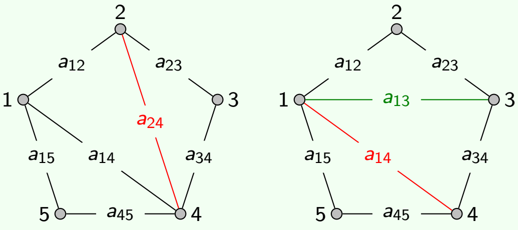
$$a_{13} = \frac{a_{12}a_{34} + a_{14}a_{23}}{a_{24}}$$

Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example



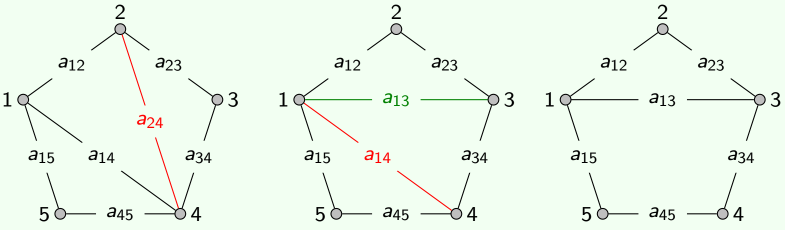
$$a_{13} = \frac{a_{12}a_{34} + a_{14}a_{23}}{a_{24}}$$

Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example



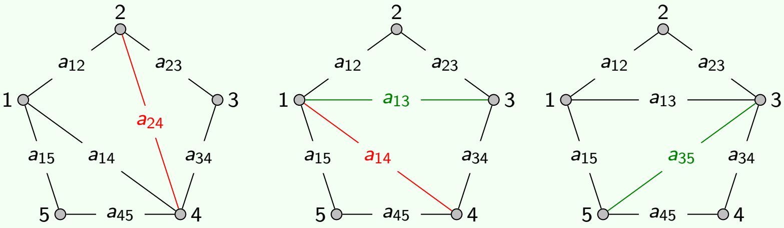
$$a_{13} = \frac{a_{12}a_{34}+a_{14}a_{23}}{a_{24}}$$

Adjacent triangulations: differ in a single element

The change of coordinates is given a simple Laurent expression:

$$\text{new variable} = \frac{\text{binomial}}{\text{old variable}}$$

Example



$$a_{13} = \frac{a_{12}a_{34} + a_{14}a_{23}}{a_{24}} \qquad a_{35} = \frac{a_{13}a_{45} + a_{15}a_{34}}{a_{14}}$$

Recap: Structures in $\mathcal{A}_n$

The commutative ring $\mathcal{A}_n$ comes with the following data.

- A special set of generators (the $\{a_{ij}\}$).
- Many special subsets of those generators (the triangulations) which 'almost' generate $\mathcal{A}_n$, in that every element can be written as a Laurent polynomial.
- A simple relation for moving between two adjacent special subsets (the Ptolemy/Plücker relations), which replaces a single element with a binomial divided by the old element.

Cluster algebras: the idea

A **cluster algebra** is a commutative ring $\mathcal{A}$ with the following data.

- A special set of generators (the cluster variables).
- Many special subsets of those generators (the clusters) which 'almost' generate $\mathcal{A}$, in that every element can be written as a Laurent polynomial.
- A simple relation for moving between two adjacent special subsets (the mutation relations), which replaces a single element with a binomial divided by the old element.

Cluster algebras frequently appear in the rings of functions on interesting spaces. Call these the **fundamental examples**.

The fundamental examples [BFZ, GSV, FG, PS...]

Cluster algebras are functions on...	Cluster variables are...
Spaces of matrices $Mat(m, n)$	Determinants of minors
Grassmannians $Gr(m, n)$	Plücker coordinates
Semisimple Lie groups G	Generalized minors
Decorated Teichmüller spaces $\widetilde{\mathcal{T}}(\Sigma)$	Lambda lengths

Our old friend $\mathcal{A}_n$ comes from $Gr(2, n)$, as well as the decorated Teichmüller space of a disc with n marked points.

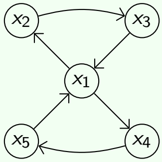
Cluster algebras are built from a **seed**, which intuitively is a single **cluster** with extra information that tells it how to **mutate**.

A seed

A seed is…

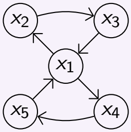
- a **cluster**: a finite set in a field (in fact, a transcendence basis over $\mathbb{Q}$);
- which is identified with the vertices of a **quiver** (a finite directed graph without loops () or 2-cycles ()),

A seed



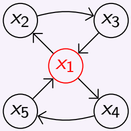
A seed can be **mutated** at any vertex.

Mutation at a vertex



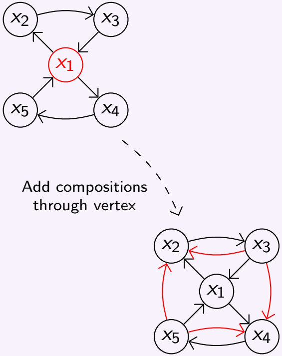
A seed can be **mutated** at any vertex.

Mutation at a vertex



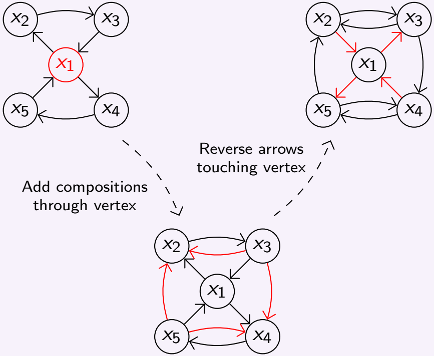
A seed can be **mutated** at any vertex.

Mutation at a vertex



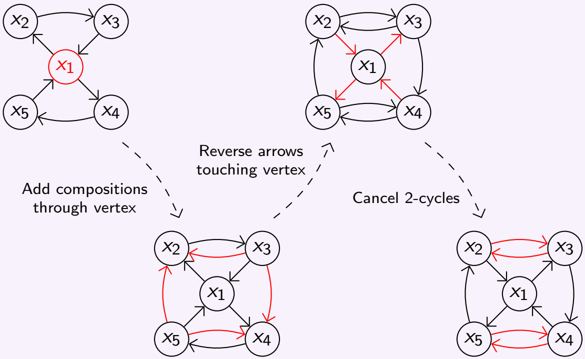
A seed can be **mutated** at any vertex.

Mutation at a vertex



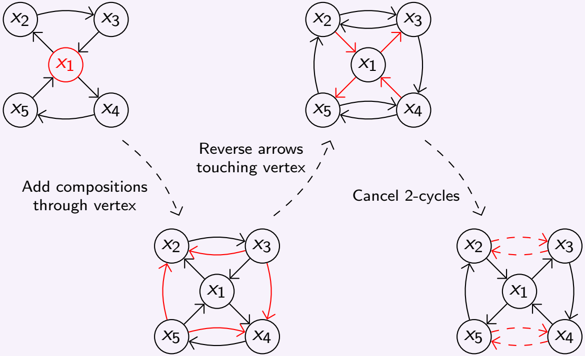
A seed can be **mutated** at any vertex.

Mutation at a vertex



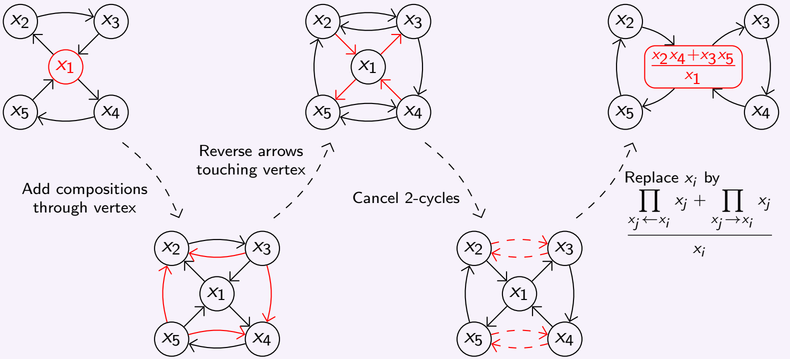
A seed can be **mutated** at any vertex.

Mutation at a vertex



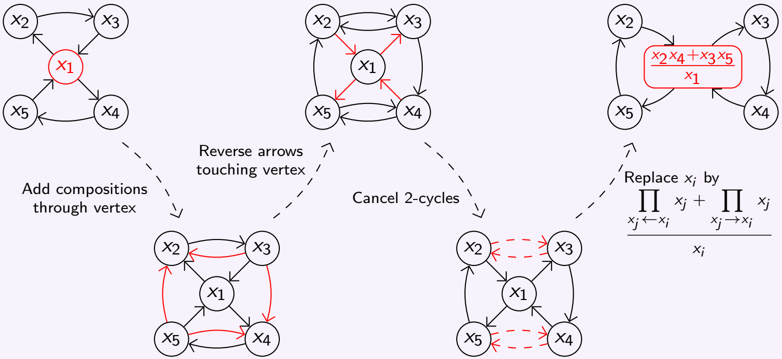
A seed can be **mutated** at any vertex.

Mutation at a vertex



A seed can be **mutated** at any vertex.

Mutation at a vertex



Two seeds connected by a sequence are **mutation-equivalent**.
Mutation-equivalent seeds live in the same field.

Given a seed, we can consider all possible sequences of mutations, and produce many clusters in the same field.

The cluster algebra of a seed

The **cluster algebra** $\mathcal{A}$ of a seed is the subring of the field generated by the union of the clusters in mutation-equivalent seeds.

A **cluster variables** in $\mathcal{A}$ is an element of one of the seeds.

Given a seed, we can consider all possible sequences of mutations, and produce many clusters in the same field.

The cluster algebra of a seed

The **cluster algebra** $\mathcal{A}$ of a seed is the subring of the field generated by the union of the clusters in mutation-equivalent seeds.

A **cluster variables** in $\mathcal{A}$ is an element of one of the seeds.

Infinite clusters

Most cluster algebras have infinitely many clusters and cluster variables, but many are still finitely generated.

This makes an arbitrary cluster algebra very difficult to study directly, without extra knowledge of a generating set.

We never required that any cluster ‘almost generates’ $\mathcal{A}$.

It’s not an axiom, but an important theorem.

Theorem (Laurent phenomenon [Fomin-Zelevinsky, 2001])

Given a cluster $C = \{x_1, x_2, ..., x_n\}$ in a cluster algebra $\mathcal{A}$, every element of $\mathcal{A}$ can be written as a Laurent polynomial in C. That is,

$$\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$$

So, an element $a \in \mathcal{A}$ can be written as a Laurent polynomial in many different ways–one for each cluster in $\mathcal{A}$.

We never required that any cluster 'almost generates' $\mathcal{A}$.

It's not an axiom, but an important theorem.

Theorem (Laurent phenomenon [Fomin-Zelevinsky, 2001])

Given a cluster $C = \{x_1, x_2, ..., x_n\}$ in a cluster algebra $\mathcal{A}$, every element of $\mathcal{A}$ can be written as a Laurent polynomial in C. That is,

$$\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$$

So, an element $a \in \mathcal{A}$ can be written as a Laurent polynomial in many different ways–one for each cluster in $\mathcal{A}$.

Question

Does this characterize the elements of $\mathcal{A}$ inside the ambient field?

We never required that any cluster 'almost generates' $\mathcal{A}$.

It's not an axiom, but an important theorem.

Theorem (Laurent phenomenon [Fomin-Zelevinsky, 2001])

Given a cluster $C = \{x_1, x_2, ..., x_n\}$ in a cluster algebra $\mathcal{A}$, every element of $\mathcal{A}$ can be written as a Laurent polynomial in C. That is,

$$\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$$

So, an element $a \in \mathcal{A}$ can be written as a Laurent polynomial in many different ways–one for each cluster in $\mathcal{A}$.

Question

Does this characterize the elements of $\mathcal{A}$ inside the ambient field?

Answer: No in general, but yes in most fundamental examples.

The set of elements satisfying the Laurent phenomenon is an interesting algebra in its own right.

The upper cluster algebra

The **upper cluster algebra** $\mathcal{U}$ of a cluster algebra $\mathcal{A}$ is

$$\mathcal{U} := \bigcap_{\substack{\text{clusters} \\ \{x_1,\ldots,x_n\}}} \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

The Laurent phenomenon is equivalent to the inclusion $\mathcal{A} \subseteq \mathcal{U}$.

The set of elements satisfying the Laurent phenomenon is an interesting algebra in its own right.

The upper cluster algebra

The **upper cluster algebra** $\mathcal{U}$ of a cluster algebra $\mathcal{A}$ is

$$\mathcal{U} := \bigcap_{\substack{\text{clusters} \\ \{x_1,\ldots,x_n\}}} \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

The Laurent phenomenon is equivalent to the inclusion $\mathcal{A} \subseteq \mathcal{U}$.

Better question

When does $\mathcal{A} = \mathcal{U}$?

There is an curious dichotomy among general cluster algebras: they are either very nice or very nasty, with no examples in between.

The following examples [BFZ '05] exemplify the general pattern.

Ex: Good behavior

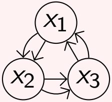
A cluster algebra is **acyclic** if it has a seed with no directed cycles.

If $\mathcal{A}$ is acyclic, then

- $\mathcal{A} = \mathcal{U}$.
- $\mathcal{A}$ is finitely generated.
- $\mathcal{A}$ is normal.
- $\mathcal{A}$ is a complete intersection.

Ex: Bad behavior

Let $\mathcal{A}$ be defined by the seed



- $\mathcal{A} \neq \mathcal{U}$.
- $\mathcal{A}$ is infinitely generated.
- $\mathcal{A}$ is not normal.†
- $\mathcal{A}$ is non-Noetherian.

Most fundamental examples are well-behaved, but are not acyclic.

Question

Is there some simple, checkable property which distinguishes the good cluster algebras from the bad ones?

Question

Is there some simple, checkable property which distinguishes the good cluster algebras from the bad ones?

I became fascinated in this in 2010 (via work on ‘skein algebras’).

Question

Is there some simple, checkable property which distinguishes the good cluster algebras from the bad ones?

I became fascinated in this in 2010 (via work on 'skein algebras').

To answer this question, I introduced **locally acyclic** cluster algebras and helped develop their properties over a series of papers.

Question

Is there some simple, checkable property which distinguishes the good cluster algebras from the bad ones?

I became fascinated in this in 2010 (via work on ‘skein algebras’).

To answer this question, I introduced **locally acyclic** cluster algebras and helped develop their properties over a series of papers.

Project goals

- Goal 1: Prove locally acyclic cluster algebras have all the properties desired of ‘good’ cluster algebras.
- Goal 2: Show that known examples of good cluster algebras are locally acyclic.

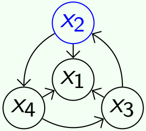
Idea: Find localizations of $\mathcal{A}$ which are acyclic cluster algebras.

Lemma [M, '12]

Let S be a subset of a seed such that every directed cycle passes through S.

Then the localization $\mathcal{A}[S^{-1}]$ is naturally an acyclic cluster algebra.

Example



A cover by localizations: the localizing sets generate 1

Locally acyclic cluster algebra (M, 2012)

A cluster algebra $\mathcal{A}$ is **locally acyclic** if it may be covered by localizations of this form.

The properties of acyclic cluster algebras listed before are local properties.†

As a direct consequence, they hold for locally acyclic cluster algebras.

† Exception

'Complete intersection' is not local, and must be weakened to locally a complete intersection.

Theorem [M, 2012]

If $\mathcal{A}$ is a locally acyclic cluster algebra, then

- $\mathcal{A} = \mathcal{U}$,
- $\mathcal{A}$ is finitely generated,
- $\mathcal{A}$ is normal, and
- $\mathcal{A}$ is locally a complete intersection.

Most fundamental examples have now been proven locally acyclic.

Theorem [M-Speyer, 2014]

The cluster algebras of **Grassmannians** are locally acyclic.
The cluster algebras of **spaces of matrices** are locally acyclic.
The cluster algebras of **SL(n)** and **GL(n)** are locally acyclic.

Theorem [M, 2012]

Given a marked surface with at least two marked points, the cluster algebra of the **decorated Teichmüller space** is locally acyclic.

Algebraic geometry: consider the complex variety

$$\mathsf{V}(\mathcal{A}) := \{\text{homomorphisms } \mathcal{A} \to \mathbb{C}\}$$

This is a complex manifold except at singularities, which local techniques are well-suited to studying.

A cluster algebra is full-rank if the skew-adjacency matrix of the quiver of any seed has non-zero determinant.

Theorem [M, 2012]

If $\mathcal{A}$ is a full-rank and locally acyclic, then $\mathsf{V}(\mathcal{A})$ is a $\mathbb{C}$-manifold.

For general locally acyclics, $V(\mathcal{A})$ may have singularities; however...

Theorem [Benito–M–Rajchgot–Smith, 2014]

If $\mathcal{A}$ is locally acyclic, $V(\mathcal{A})$ has (at worst) canonical singularities.

A variety has canonical singularities if it has a resolution of singularities with non-negative discrepancies.

Proof via reduction to positive characteristic

For a field k with $\mathrm{char}(k) = p > 2$, the algebra $k \otimes \mathcal{A}$ has a Frobenius endomorphism

$$a \mapsto a^p$$

We prove this has a splitting which does not split any non-trivial ideals. This implies $k \otimes \mathcal{A}$ lacks 'bad' singularities for all k.

With the main goals of the program accomplished, what's next?

Gross, Hacking, Keel and Kontsevich recently defined a conjectural basis of theta functions in any cluster algebra (via MS for log CYs).

Problem

The construction involves formal series which may not converge.

Question

Do these formal series converge for locally acyclic cluster algebras?

A positive answer would provide a canonical basis in the coordinate ring of many important varieties.