The twist for positroid varieties

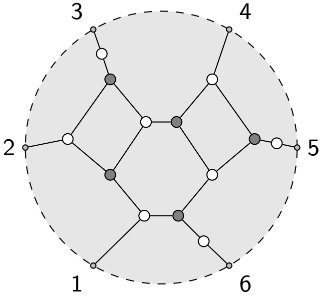
Greg Muller, joint with David Speyer

July 7, 2016

2-colored graphs in the disc

Let G be a graph in the disc with...

- a bipartite 2-coloring of its internal vertices, and
- a clockwise indexing of its boundary vertices from 1 to n.

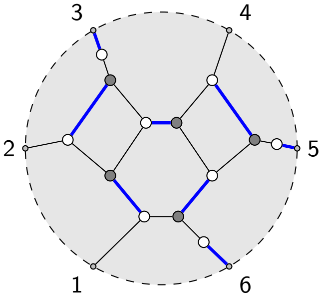


Assume (for simplicity)

Each boundary vertex is adjacent to one vertex, which is white.

Matchings of G

For this talk, a **matching** of G is a subset of the edges for which every internal vertex is in exactly one edge.



Easy observation

Every matching of G contains

$$k := |\text{white vertices}| - |\text{black vertices}|$$

boundary vertices.

Two maps, both alike in dignity

In an unpublished note in 2003, Postnikov associated two maps to such a graph in the disc, which may be defined using matchings.

- A boundary measurement map

$$\mathbb{B} : \text{an algebraic torus} \longrightarrow \text{a positroid variety}$$

- A cluster: a rational map

$$\mathbb{F} : \text{a positroid variety} \dashrightarrow \text{an algebraic torus}$$

These maps have proven fruitful in studying flows in networks, total positivity, elementary factorizations of matrices, integrable systems and perturbative field theories.

Despite abundant applications, basic questions about these maps remained open for more than a decade; like, how are they related?

Weighted enumeration of matchings

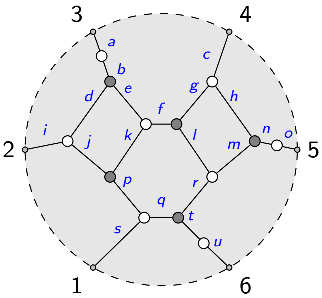
Question

What are the matchings of G with boundary $I \subset \{1, 2, \ldots, n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

$$\text{an edge weighting } w \mapsto \begin{array}{c}\text{total weight of matchings}\\ \text{with boundary } I\end{array}$$



$$\mathcal{Z}_{356}(w) =$$

Weighted enumeration of matchings

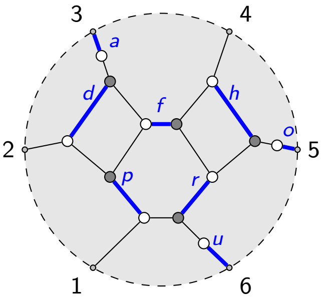
Question

What are the matchings of G with boundary $I \subset \{1, 2, \dots, n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

an edge weighting $w \mapsto$ total weight of matchings with boundary I



$$\mathcal{Z}_{356}(w) = adfprou + \dots$$

Weighted enumeration of matchings

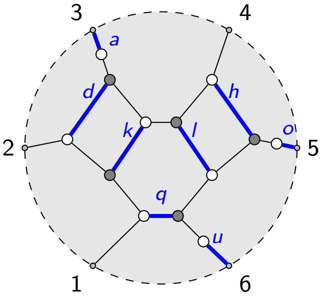
Question

What are the matchings of G with boundary $I \subset \{1, 2, \dots, n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

an edge weighting $w \mapsto$ total weight of matchings with boundary I



$adfprou +$

$adhklqou + \dots$

$\mathcal{Z}_{356}(w) =$

Weighted enumeration of matchings

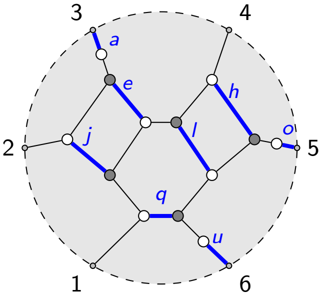
Question

What are the matchings of G with boundary $I \subset \{1, 2, \ldots, n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

an edge weighting $w \mapsto$ total weight of matchings with boundary I



$$\mathcal{Z}_{356}(w) = aehjlqou + adhklqou + adfprou + \ldots$$

Weighted enumeration of matchings

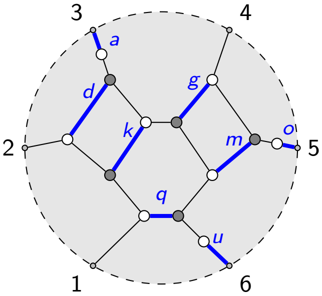
Question

What are the matchings of G with boundary $I \subset \{1, 2, ..., n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

an edge weighting $w \mapsto$ total weight of matchings with boundary I



$$\mathcal{Z}_{356}(w) = \begin{aligned} &adfprou + \\ &adhklqou + \\ &aehjlqou + \\ &adgkmqou + ... \end{aligned}$$

Weighted enumeration of matchings

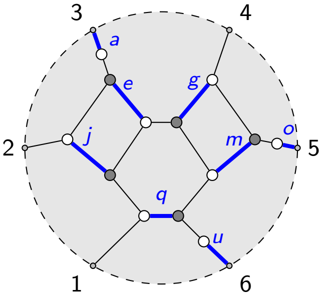
Question

What are the matchings of G with boundary $I \subset \{1, 2, \ldots, n\}$?

We encode the answer in the **partition function $\mathcal{Z}_I$** of I.

$$\mathcal{Z}_I : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow \mathbb{C}$$

$$\text{an edge weighting } w \mapsto \begin{array}{c}\text{total weight of matchings}\\ \text{with boundary } I\end{array}$$



$$\mathcal{Z}_{356}(w) = \begin{array}{l} adfprou\, + \\ adhklqou\, + \\ aehjlqou\, + \\ adgkmqou\, + \\ aegjmqou \end{array}$$

The Plücker relations

The partition functions satisfy a surprising family of identities.

Theorem (Kuo '04, Postnikov '06)

The partition functions $\{\mathcal{Z}_I\}$ satisfy the **Plücker relations**.

Concretely, this means $\exists$ a $k \times n$ matrix A_w such that $\forall I$,

$$\mathcal{Z}_I(w) = \Delta_I(A_w) := \text{determinant of columns of } A_w \text{ in } I$$

$$A_w = \begin{bmatrix} 1 & 0 & -\frac{aep}{bks} & 0 & \frac{fmop}{klns} & \frac{klqu+fpru}{klst} \\ 0 & 1 & \frac{adk+aej}{bik} & 0 & -\frac{fjmo}{ikln} & -\frac{fjru}{iklt} \\ 0 & 0 & 0 & bciklnst & bikost(hl+gm) & bgiknrsu \end{bmatrix}$$

The Plücker relations

The partition functions satisfy a surprising family of identities.

Theorem (Kuo '04, Postnikov '06)

The partition functions $\{\mathcal{Z}_I\}$ satisfy the **Plücker relations**.

Concretely, this means $\exists$ a $k \times n$ matrix A_w such that $\forall I$,

$$\mathcal{Z}_I(w) = \Delta_I(A_w) := \text{determinant of columns of } A_w \text{ in } I$$

$$A_w = \begin{bmatrix} 1 & 0 & -\frac{aep}{bks} & 0 & \frac{fmop}{klns} & \frac{klqu+fpru}{klst} \\ 0 & 1 & \frac{adk+aej}{bik} & 0 & -\frac{fjmo}{ikln} & -\frac{fjru}{iklt} \\ 0 & 0 & 0 & bciklnst & bikost(hl+gm) & bgiknrsu \end{bmatrix}$$

The Plücker relations

The partition functions satisfy a surprising family of identities.

Theorem (Kuo '04, Postnikov '06)

The partition functions $\{\mathcal{Z}_I\}$ satisfy the **Plücker relations**.

Concretely, this means $\exists$ a $k \times n$ matrix A_w such that $\forall I$,

$$\mathcal{Z}_I(w) = \Delta_I(A_w) := \text{determinant of columns of } A_w \text{ in } I$$

$$A_w = \begin{bmatrix} 1 & 0 & -\frac{aep}{bks} & 0 & \frac{fmop}{klns} & \frac{klqu+fpru}{klst} \\ 0 & 1 & \frac{adk+aej}{bik} & 0 & -\frac{fjmo}{ikln} & -\frac{fjru}{iklt} \\ 0 & 0 & 0 & bciklnst & bikost(hl+gm) & bgiknrsu \end{bmatrix}$$

$$\Delta_{356}(A_w) = \begin{matrix} adfprou + adhklqou + aehjlqou \\ + adgkmqou + aegjmqou \end{matrix} = \mathcal{Z}_I(w)$$

The boundary measurement map

While A_w is not uniquely determined, the span of its rows is!

The $\mathcal{Z}_I(w)$ are encoded (up to scaling) in the well-defined subspace

$$\mathbb{B}(w) := \mathrm{rowspan}(A_w) \subset \mathbb{C}^n$$

We get a map to $Gr(k,n)$, the **Grassmannian** of k-spaces in $\mathbb{C}^n$.

The boundary measurement map (raw version)

$$\mathbb{B} : (\mathbb{C}^\times)^{\mathrm{Edges}(G)} \longrightarrow Gr(k,n)$$

an edge weighting $w \mapsto \mathrm{rowspan}(A_w)$

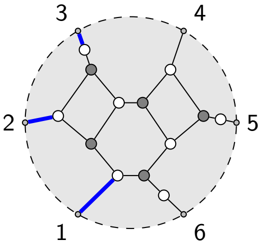
Restricting the target

The image of $\mathbb{B}$ is typically much smaller than $Gr(k,n)$, for the simple reason that some of the $\mathcal{Z}_I$ can be zero.

The positroid variety of G

Let $\Pi(G) \subset Gr(k,n)$ parametrize rowspans of matrices A with

$$\Delta_I(A) = 0 \text{ whenever } G \text{ has no matchings with boundary } I$$



Every I except 123 is the boundary of at least one matching in G on the left.

$$\Pi(G) = \mathrm{rowspan} \begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Quotienting the domain

$\mathbb{B}$ also factors through a simple symmetry of the edge weights.

Gauge transformations

A gauge transformation at an internal vertex v multiplies the weight of each edge adjacent to v by some fixed scalar $\lambda \in \mathbb{C}^\times$.

These transformations do not change the subspace $\mathbb{B}(w)$.

The boundary measurement map (refined version)

$$\mathbb{B} : (\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \longrightarrow \Pi(G)$$

Reduced graphs

The map $\mathbb{B}$ is most interesting for reduced graphs.

Reduced graphs

The graph G is **reduced** if every face is a disc, and there is no G' with fewer faces and $\Pi(G) = \Pi(G')$.

Intuitively, these are the graphs which realize $\Pi(G)$ most efficiently.

Expectation

When G is reduced, the map

$$\mathbb{B} : (\mathbb{C}^{\times})^{\mathrm{Edges}(G)}/\mathrm{Gauge} \longrightarrow \Pi(G)$$

is an open inclusion of varieties.

Applications are particularly interested in a constructive answer: given $[V]$ in $\Pi(G)$, construct w such that $\mathbb{B}(w) = V$.

To attack this problem, explore the geometry of $\Pi(G)$ and construct our second map of interest, we ask...

Question

How can we describe a point in $\Pi(G)$?

Coordinates on $\Pi(G)$

Plücker coordinates

Given a matrix A, $\mathrm{rowspan}(A)$ is determined by the Plücker coordinates $\{\Delta_I(A)\}$, running over all $I \subset \{1, 2, ..., n\}$ of size k.

However, there is a large amount of redundancy in these numbers.

Observation

If G is reduced, the dimension of $\Pi(G)$ is $|\mathrm{Faces}(G)| - 1$.

So, at least $|\mathrm{Faces}(G)|$-many Plücker coordinates are needed to determine a point in $\Pi(G)$.

Question

Can we determine a point in $\Pi(G)$ with **exactly** $|\mathrm{Faces}(G)|$-many Plücker coordinates?

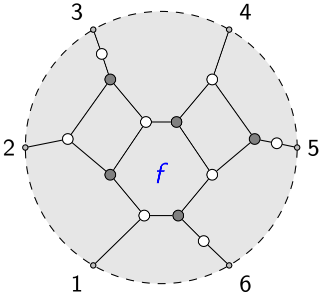
Matchings associated to faces

Idea: use boundaries of distinguished matchings to choose the Δ_I.

Lemma (M-Speyer)

Given reduced G and a face f, there $\exists!$ matching M_f containing

1. every edge in ∂f going clockwise from white to black, and
2. one fewer than half the edges around every other face.



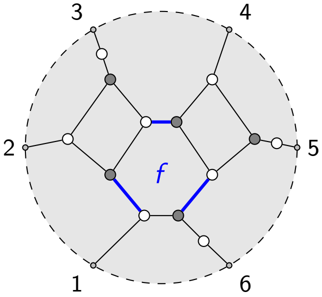
Matchings associated to faces

Idea: use boundaries of distinguished matchings to choose the Δ_I.

Lemma (M-Speyer)

Given reduced G and a face f, there $\exists!$ matching M_f containing

1. every edge in ∂f going clockwise from white to black, and
2. one fewer than half the edges around every other face.



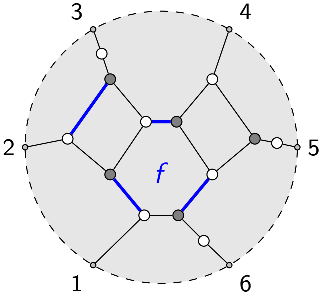
Matchings associated to faces

Idea: use boundaries of distinguished matchings to choose the Δ_I.

Lemma (M-Speyer)

Given reduced G and a face f, there $\exists!$ matching M_f containing

1. every edge in ∂f going clockwise from white to black, and
2. one fewer than half the edges around every other face.



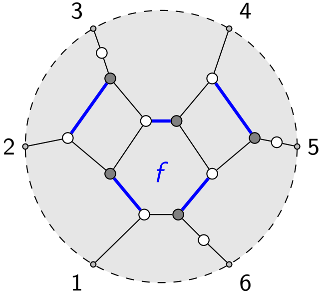
Matchings associated to faces

Idea: use boundaries of distinguished matchings to choose the Δ_I.

Lemma (M-Speyer)

Given reduced G and a face f, there $\exists!$ matching M_f containing

1. every edge in ∂f going clockwise from white to black, and
2. one fewer than half the edges around every other face.



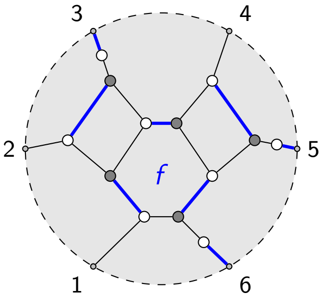
Matchings associated to faces

Idea: use boundaries of distinguished matchings to choose the Δ_I.

Lemma (M-Speyer)

Given reduced G and a face f, there $\exists!$ matching M_f containing

1. every edge in ∂f going clockwise from white to black, and
2. one fewer than half the edges around every other face.



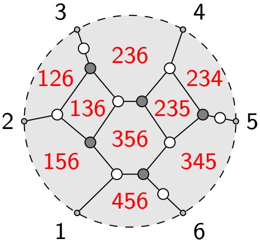
Plücker coordinates associated to faces

Each face determines a Plücker coordinate.

$$\Delta_f := \Delta_{\partial M_f}$$

We combine these into a map on $\Pi(G)$.

$$\mathbb{F} : \Pi(G) \longrightarrow \mathbb{C}^{\mathrm{Faces}(G)}/\mathrm{Scaling}$$



Expectation

$\mathbb{F}$ restricts to an isomorphism on the subset where no Δ_f is zero.

$$\mathbb{F} : \Pi(G) \dashrightarrow (\mathbb{C}^{\times})^{\mathrm{Faces}(G)}/\mathrm{Scaling}$$

This is a consequence of a deeper conjecture (which we'll ignore).

Conjecture [Scott, Postnikov, M-Speyer, LeClerc]

The homogeneous coordinate ring of $\Pi(G)$ is a cluster algebra, and the Plücker coordinates of the faces of G form a cluster.

Two conjectural tori associated to a reduced graph

So, a reduced graph G determines two maps, and each map conjecturally defines an open subset in $\Pi(G)$.

- The image of the boundary measurement map

$$\mathbb{B} : (\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \longrightarrow \Pi(G)$$

- The domain of definition of the cluster of Plücker coordinates

$$\mathbb{F} : \Pi(G) \dashrightarrow (\mathbb{C}^\times)^{\mathrm{Faces}(G)}/\mathrm{Scaling}$$

Natural question

What is the relation between these two subsets?

In a simple world, they'd coincide and we'd have an isomorphism

$$\mathbb{F} \circ \mathbb{B} : (\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \xrightarrow{\sim} (\mathbb{C}^\times)^{\mathrm{Faces}(G)}/\mathrm{Scaling}$$

The need for a twist

In the real world, we need a **twist** automorphism τ of $\Pi(G)$, which will fit into a composite isomorphism

$$\begin{tikzcd}
(\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \arrow[dr, "\mathbb{B}"'] & & & (\mathbb{C}^\times)^{\mathrm{Faces}(G)}/\mathrm{Scaling} \\
& \Pi(G) \arrow[r, "\tau"] & \Pi(G) \arrow[ur, dashed, "\mathbb{F}"'] &
\end{tikzcd}$$

and thus take the image of $\mathbb{B}$ to the domain of $\mathbb{F}$.

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

$$\begin{bmatrix} 1 & \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 1 & 2 \\ 0 & \color{red}{0} & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & ? & ? & ? \\ -1 & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, \ldots, A_{j-1}\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & ? & ? & ? \\ -1 & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

$$\begin{bmatrix} 1 & 1 & 0 & \color{red}{0} \\ 0 & 1 & 1 & \color{red}{2} \\ 0 & 0 & 0 & \color{red}{1} \end{bmatrix} \mapsto \begin{bmatrix} 1 & ? & ? & ? \\ -1 & ? & ? & ? \\ 2 & ? & ? & ? \end{bmatrix}$$

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition: The twist

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

Properties of the twist

Lemma (M-Speyer)

The twist has an inverse given by the same formula as τ, except moving to the left instead of the right.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem (M-Speyer)

The twist map descends to a piecewise-regular automorphism

$$\tau : \Pi(G) \longrightarrow \Pi(G)$$

The twist makes everything work!

Theorem (M-Speyer)

If G is reduced, there is a combinatorially-defined isomorphism such that the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \arrow[r, "\sim"] \arrow[d, "\mathbb{B}"'] & (\mathbb{C}^\times)^{\mathrm{Faces}(G)}/\mathrm{Scaling} \\
\Pi(G) \arrow[r, "\tau"] & \Pi(G) \arrow[u, dashed, "\mathbb{F}"']
\end{tikzcd}$$

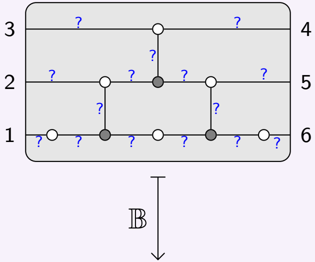
Corollaries!

- The boundary measurement map $\mathbb{B}$ is an open inclusion.
- The cluster $\mathbb{F}$ is an isomorphism on its domain.

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell



$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

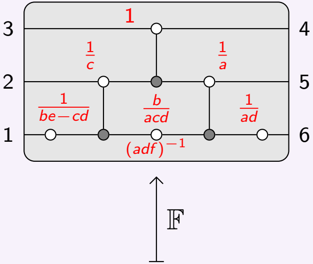
Example: The unipotent cell in $GL(3)$, as a positroid cell

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell

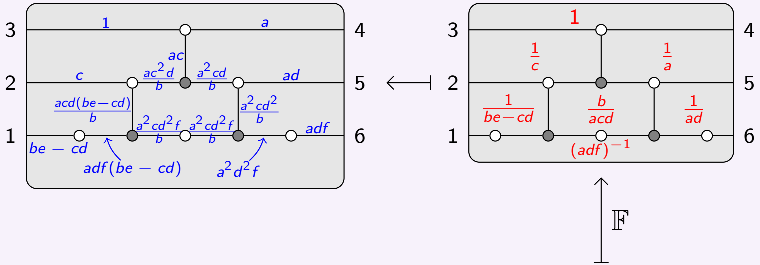


$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell

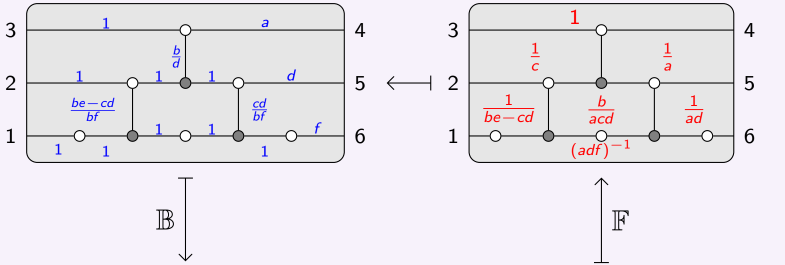


$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

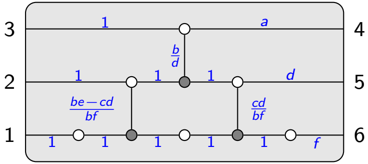
Example: The unipotent cell in $GL(3)$, as a positroid cell



$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Relation to the Chamber Ansatz

We have the following boundary measurement map computation.



$$\xmapsto{\mathbb{B}} \begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

This is equivalent to a factorization into elementary matrices.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{be-cd}{bf} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{d} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{cd}{bf} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix}$$