The geometry of cluster algebras

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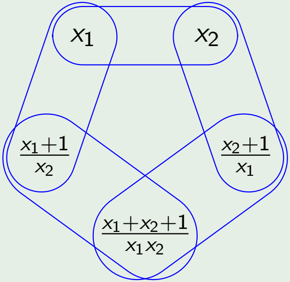
Cluster algebras (the idea)

A **cluster algebra** is a commutative ring generated by distinguished elements called **cluster variables**.

The set of cluster variables can be recursively generated from distinguished subsets called **clusters**, by a process called **mutation**.

Ex: The cluster algebra $\mathcal{A}(A_2)$



- The five cluster variables are rational functions in x_1 and x_2.
- The five clusters are circled in blue.
- The cluster algebra is the subring of $\mathbb{Q}(x_1, x_2)$ generated by

$$x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}$$

Motivating examples of cluster algebras

Cluster algebras occur in the rings of functions of many spaces; we call them the motivating examples.†

†Historical note

The actual motivation is from Lusztig's canonical bases for $\mathcal{U}_q\mathfrak{n}$.

In each case, well-known functions are cluster variables, and mutation encodes standard identities between these functions.

The motivating examples

Cluster algebras are functions on...	Cluster variables are...
Spaces of matrices $Mat(m, n)$	Determinants of minors
Grassmannians $Gr(m, n)$	Plücker coordinates
Semisimple Lie groups G	Generalized minors
Decorated Teichmüller spaces $\widetilde{\mathcal{T}}(\Sigma)$	Lambda lengths

Basic ingredient: a seed

Disclaimer

While there are many minor variations on the definition of a cluster algebra, this talk will define cluster algebras coming from by quivers.

A **seed** of rank n is a transcendence basis

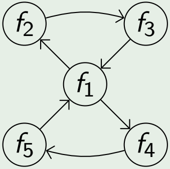
$$f_1, f_2, \dots, f_n \in \mathbb{Q}(x_1, \dots, x_n)$$

which is identified with the vertices of a **quiver**† (ie, a finite directed graph).

†Restriction

The quiver can't have loops () or 2-cycles ().

Ex: A seed



Mutation of a seed

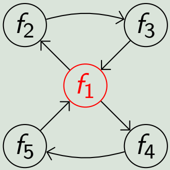
A seed is **mutated** at a specified vertex in the following 4 steps.

The steps of mutation

1. Add compositions of composible pairs of arrows at f_i.
2. Reverse arrows incident to f_i.
3. Cancel 2-cycles.
4. Replace f_i with

$$\left(\prod_{\substack{\text{arrows} \\ f_i \to f_j}} f_j + \prod_{\substack{\text{arrows} \\ f_i \leftarrow f_j}} f_j \right) \Big/ f_i$$

Ex: Mutation at f_1



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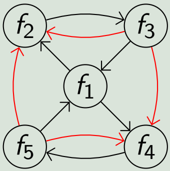
Ex: Step 1



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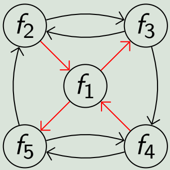
Ex: Step 2



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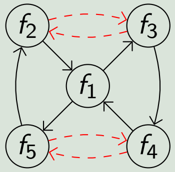
Ex: Step 3



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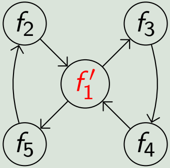
Ex: Step 4



$$f_1' := \frac{f_3 f_5 + f_2 f_4}{f_1}$$

Cluster algebras (the definition)

Some easy observations.

- The mutation of a seed is a seed, so mutation can be iterated.
- Mutation at the same vertex twice in a row returns to the original seed. Being related by a sequence of mutations is an equivalence relation on seeds, called mutation-equivalence.

Definition: Cluster algebra [Fomin–Zelevinsky, '01]

The **cluster algebra** $\mathcal{A}$ of a seed S is the ring in $\mathbb{Q}(x_1, ..., x_n)$ generated by all functions in seeds mutation-equivalent to S.

A **cluster variable** is a function in any mutation-equivalent seed, and a **cluster** is n cluster variables which are in a seed together.

The Laurent phenomenon

Theorem (The Laurent phenomenon [Fomin–Zelevinsky, '01])

For every cluster $\{f_1, ..., f_n\}$,

$$\mathcal{A} \subset \mathbb{Z}[f_1^{\pm 1}, ..., f_n^{\pm 1}] \subset \mathbb{Q}(x_1, ..., x_n)$$

Therefore, an element $a \in \mathcal{A}$ can be written as a Laurent polynomial in many different ways – one for each cluster in $\mathcal{A}$.

Question

Does this property characterize elements of $\mathcal{A}$ inside $\mathbb{Q}(x_1, ..., x_n)$?

Answer: No in general; but yes in most motivating examples.

The upper cluster algebra

The set of rational functions in $\mathbb{Q}(x_1, ..., x_n)$ with all of these Laurent expansions is an interesting algebra in its own right.

Definition: Upper cluster algebra [BFZ, '05]

The **upper cluster algebra** $\mathcal{U}$ of $\mathcal{A}$ is defined as

$$\mathcal{U} := \bigcap_{\substack{\text{clusters} \\ \{f_1,...,f_n\}}} \mathbb{Z}[f_1^{\pm 1}, ..., f_n^{\pm 1}] \subset \mathbb{Q}(x_1, ..., x_n)$$

The Laurent phenomenon is equivalent to the inclusion $\mathcal{A} \subseteq \mathcal{U}$.

Better question

When does $\mathcal{A} = \mathcal{U}$?

The good, the bad, and the unknown

A curious dichotomy has been observed in cluster algebras: they are either very nice or very nasty, with no examples in between.

The following two examples exemplify the general pattern.

Ex: Good behavior

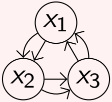
A cluster algebra is **acyclic** if it has a seed with no directed cycles.

If $\mathcal{A}$ is acyclic, then

- $\mathcal{A} = \mathcal{U}$.
- $\mathcal{A}$ is finitely generated.
- $\mathcal{A}$ is integrally closed.
- $\mathcal{A}$ is a complete intersection.

Ex: Bad behavior

Let $\mathcal{A}$ be defined by the seed



- $\mathcal{A} \neq \mathcal{U}$.
- $\mathcal{A}$ is infinitely generated.
- $\mathcal{A}$ is not $S2$.
- $\mathcal{A}$ is non-Noetherian.

Most motivating examples are well-behaved, but are not acyclic.

Question

Is there a unifying explanation for why these examples are nice?

A conjectural explanation: these cluster algebras are locally acyclic.

From algebra to geometry

For a unital commutative ring R, define

$$\mathrm{V}(R) := Hom(R, \mathbb{C}) = \{\text{unital ring maps } R \to \mathbb{C}\}$$

The set $\mathrm{V}(R)$ has a natural topology.

For algebraic geometers

$\mathrm{V}(R)$ is the set of $\mathbb{C}$-points of the affine scheme $Spec(R)$.

Algebraic properties of R become geometric properties of $\mathrm{V}(R)$.

Ex: Laurent inclusions become sub-tori

For a cluster $\{f_1 ... f_n\}$, the inclusion $\mathcal{A} \subset \mathbb{Z}[f_1^{\pm 1}, ..., f_n^{\pm 1}]$ becomes

$$\mathrm{V}(\mathcal{A}) \supset \mathrm{V}(\mathbb{Z}[f_1^{\pm 1}, ..., f_n^{\pm 1}]) \cong (\mathbb{C}^\times)^n$$

Thus, each cluster defines an algebraic torus $(\mathbb{C}^\times)^n$ inside $\mathrm{V}(\mathcal{A})$.

Locally acyclic cluster algebras

Idea: Cover $\mathrm{V}(\mathcal{A})$ using simpler cluster algebras.

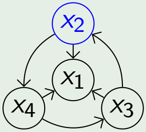
Lemma (M, '12)

Let $S \subset \{f_1, \ldots, f_n\}$ be a subset of a cluster, such that every directed cycle in the seed passes through a vertex in S.

Then the localization $\mathcal{A}[S^{-1}]$ is naturally an acyclic cluster algebra.

Call $\mathrm{V}(\mathcal{A}[S^{-1}]) \subseteq \mathrm{V}(\mathcal{A})$ an acyclic chart.

Ex: An acyclic chart



$\mathrm{V}(\mathcal{A}[x_2^{-1}]) \subset \mathrm{V}(\mathcal{A})$ is an acyclic chart

Definition: Locally acyclic cluster algebra (M '12)

A cluster algebra $\mathcal{A}$ is **locally acyclic** if $\mathrm{V}(\mathcal{A})$ can be covered by finitely many acyclic charts.

Local acyclic cluster algebras: The motivating examples

A cluster algebra can be shown to be locally acyclic by a simple combinatorial method I call the Banff algorithm.

In the one class of examples, a (nearly) complete answer is known.

Theorem (M '12)

- **Decorated Teichmüller spaces:** *For Σ a marked surface with at least two marked points, $\widetilde{\mathcal{T}}(\Sigma)$ is locally acyclic.*

Other simple motivating examples can be checked by hand.

Proposition

- **Matrices:** *For $m, n \leq 5$, $Mat^{\circ}(m, n)$ is locally acyclic.*
- **Grassmannians:** *For $m \leq 3$, $Gr^{\circ}(m, n)$ is locally acyclic.*
- **Lie groups:** *For $n \leq 5$, $SL^{\circ}(n)$ is locally acyclic.*

Locally acyclic cluster algebras: First properties

The properties of acyclic cluster algebras listed before are local properties: they can be checked locally in $V(\mathcal{A})$.†

As a direct consequence, they hold for locally acyclic cluster algebras.

† Exception

'Complete intersection' is not local, and must be weakened to locally a complete intersection.

Theorem (M, '12)

If $\mathcal{A}$ is a locally acyclic cluster algebra, then

- $\mathcal{A} = \mathcal{U}$,
- $\mathcal{A}$ *is finitely generated,*
- $\mathcal{A}$ *is integrally closed, and*
- $\mathcal{A}$ *is locally a complete intersection.*

Locally acyclic cluster algebras: Local geometry

This local approach is well-suited to studying singularities in $\mathrm{V}(\mathcal{A})$.

A cluster algebra is full-rank if the skew-adjacency matrix of the quiver in any seed is full-rank.

Theorem (M, '12)

If $\mathcal{A}$ is a full-rank and locally acyclic, then $\mathrm{V}(\mathcal{A})$ is a $\mathbb{C}$-manifold.

For general locally acyclics, $\mathrm{V}(\mathcal{A})$ may have singularities; however...

Theorem (M–Rajchgot–Smith, in preparation)

If $\mathcal{A}$ is locally acyclic, $\mathrm{V}(\mathcal{A})$ has (at worst) rational singularities.

This is proven by means of a stronger result. For a field k with $char(\mathrm{k}) > 2$, the algebra $\mathrm{k} \otimes_{\mathbb{Z}} \mathcal{A}$ has a regular F-splitting.

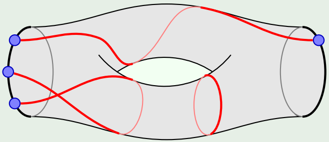
Those cluster algebras coming from decorated Teichmüller spaces have a surprising and powerful connection to skein algebras: diagrammatic algebras which arose in the study of knot invariants.

Curves in marked surfaces

A **marked surface** Σ is an oriented surface with boundary, with a finite set of marked points in $\partial\Sigma$.

A **curve** in a marked surface is a connected, immersed curve, with any endpoints at marked points.

Ex: Curves



Curves are considered up to endpoint-fixed, immersed homotopy.

Internal marked points

Marked surfaces, curves and skein algebras can be defined with internal marked points, but the connection to cluster algebras is more complicated.

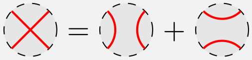
The (semi-classical) skein algebra

Fix a marked surface Σ. Let $\mathbf{Links}(\Sigma)$ be the commutative ring of $\mathbb{Z}$-combinations of finite sets of curves in Σ (up to homotopy).

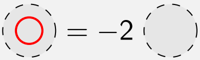
Definition: Skein algebra of a marked surface (M, '12)

The (semi-classical) **skein algebra** $\mathbf{Sk}_1(\Sigma)$ is the quotient of $\mathbf{Links}(\Sigma)$ which imposes the following three classes of relations.

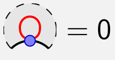
Kauffman skein relation



Contractible loop



Contractible arc



In each case, the dashed circle is a small neighborhood in Σ, and the curves remain unchanged outside this neighborhood.

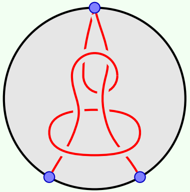
This definition generalizes the skein algebra for unmarked Σ, a central object of study in knot theory for the last two decades.

Why is $\mathbf{Sk}_1(\Sigma)$ interesting?

There is a quantum skein algebra $\mathbf{Sk}_q(\Sigma)$, which keeps track of how intersecting curves pass over or under each other. $\mathbf{Sk}_1(\Sigma)$ is the $q = 1$ specialization of $\mathbf{Sk}_q(\Sigma)$.

For unmarked Σ, $\mathbf{Sk}_q(\Sigma)$ was introduced independently by Turaev and Przytycki to compute the Jones polynomial of a knot.

Ex: Curves in $\mathbf{Sk}_q(\Sigma)$



For unmarked Σ, $\mathbf{Sk}_1(\Sigma)$ is the algebra of polynomial invariants of twisted $SL_2(\mathbb{C})$-local systems on Σ [Barrett, '97].

Theorem (M–Samuelson–D.Thurston, in preparation)

For general Σ, $\mathbf{Sk}_1(\Sigma)$ is the algebra of polynomial invariants of twisted, decorated $SL_2(\mathbb{C})$-local systems on Σ.

Cluster algebras of marked surfaces

Σ is triangulatable if there is a set of non-intersecting curves in Σ whose complement is a union of triangles.

Enough marked points

Adding enough marked points makes any Σ triangulatable.

Theorem (Gekhtman–Shapiro–Vainshtein, '05)

For triangulatable Σ, there is a cluster algebra $\mathcal{A}(\Sigma)$ such that...

- *an arc without self-intersections is a cluster variable,*
- *the set of curves in a triangulation is a cluster, and*
- *every element of $\mathcal{A}(\Sigma)$ is naturally a function on $\widetilde{\mathcal{T}}(\Sigma)$, the decorated Teichmüller space of Σ.*

Theorem (M, '12)

If Σ has at least two marked points, $\mathcal{A}(\Sigma)$ is locally acyclic.

Skein algebras and cluster algebras

A boundary curve in Σ is a curve entirely contained in the boundary $\partial\Sigma$. There are always finitely many boundary curves.

Theorem (M, '12)

Let Σ be triangulatable with at least two marked points, and let $\{c_1, c_2, ..., c_m\}$ be the set of boundary curves in Σ. Then there is a natural isomorphism

$$\mathcal{A}(\Sigma) \xrightarrow{\sim} \mathbf{Sk}_1(\Sigma)[c_1^{-1}, c_2^{-1}, ..., c_m^{-1}]$$

Proof by quantization

The proof is by means of a more general result. There is a notion of a quantum cluster algebra $\mathcal{A}_q$, such that the $q = 1$ specialization $\mathcal{A}_1$ is a cluster algebra. Then there is a quantum cluster algebra $\mathcal{A}_q(\Sigma)$ and a natural isomorphism

$$\mathcal{A}_q(\Sigma) \xrightarrow{\sim} \mathbf{Sk}_q(\Sigma)[c_1^{-1}, c_2^{-1}, ..., c_m^{-1}]$$

The key lemma in the proof is a quantum version of local acyclicity.

Consequences for the skein algebra

Here are some corollaries for $\mathbf{Sk}_1(\Sigma)$ (for Σ as before).

- Each triangulation determines an embedding of the form
$$\mathbf{Sk}_1(\Sigma) \subset \mathbb{Z}[f_1^{\pm 1}, ..., f_n^{\pm 1}]$$
- $\mathbf{Sk}_1(\Sigma)[c_1^{-1}, ..., c_n^{-1}]$ is finitely-generated and integrally closed.
- $\mathrm{V}(\mathbf{Sk}_1(\Sigma)[c_1^{-1}, ..., c_n^{-1}]) \subset \mathrm{V}(\mathbf{Sk}_1(\Sigma))$ is a $\mathbb{C}$-manifold.

A conjectural stratification of $\mathrm{V}(\mathbf{Sk}_1(\Sigma))$

Conjecturally, $\mathrm{V}(\mathbf{Sk}_1(\Sigma)[c_1^{-1}, ..., c_n^{-1}])$ is the open stratum in a stratification of $\mathrm{V}(\mathbf{Sk}_1(\Sigma))$, where each stratum is $\mathrm{V}(\mathcal{A})$ of some cluster algebra.

Consequences for the cluster algebra

The isomorphism gives $\mathcal{A}(\Sigma)$ a diagrammatic realization which is usually easier to work with than the defining construction.

This enables many conjectures for cluster algebras to be verified in the case of marked surfaces, such as the following.

Ex: Strongly positive bases

A **strongly positive basis** for $\mathcal{A}$ is a $\mathbb{Z}$-basis where the product of two basis elements is a positive combination of basis elements.

Conjecture (Fomin–Zelevinsky, '01)

Every cluster algebra has a strongly positive basis.

Theorem (Musiker–Schiffler–Williams, '12)

Let Σ be as above. Then $\mathcal{A}(\Sigma)$ has a strongly positive basis.

Future directions

- Proving the local acyclicity of classes of motivating examples.
- The stratification of $V(\mathbf{Sk}_1(\Sigma))$, and its connection to...
  - homogeneous Poisson prime ideals in $\mathbf{Sk}_1(\Sigma)$,
  - compatibly F-split ideals in $\mathrm{k} \otimes \mathbf{Sk}_1(\Sigma)$ (for $char(\mathrm{k}) > 2$), and
  - homogeneous completely prime ideals in $\mathbf{Sk}_q(\Sigma)$.
- Generalizing topological constructs from $\mathbf{Sk}_1(\Sigma)$, such as loop elements, to other cluster algebras.
- Defining the quantum skein algebra $\mathbf{Sk}_1(\Sigma)$ for marked surfaces with internal marked points.

Thank you for listening.

-G