Linear Algebra, Spring 2016
Worksheet, April 14

Core Concepts:

- Vector Spaces
  - span, linear independence, basis, dimension, subspaces
- Linear Transformations
  - kernel, range, isomorphism
- Solving Systems of Linear Equations
  - homogeneous systems, inhomogeneous systems

The following statements explore the relationships between the core concepts and other topics from the class. Try to prove each statement.

1. If $\dim V = n$, then the following statements are equivalent
   (a) $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of V
   (b) $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent
   (c) $\vec{v}_1, \ldots, \vec{v}_n$ span V
2. Let $L : V \to W$ be a linear transformation. Then $\operatorname{Ker} L$ is a subspace of V, $\operatorname{Range} L$ is a subspace of W.
3. If U is a subspace of V, then $L(U) = \{ L(\vec{u}) \mid \vec{u} \in U \}$ is a subspace of W, and $\dim L(U) \leq \dim U$. If L is an isomorphism then $\dim L(U) = \dim U$.
4. If U is a subspace of W, then $L^{-1}(U) = \{ \vec{v} \in V \mid L(\vec{v}) \in U \}$ is a subspace of V. If U is contained in the range of L, then $\dim L^{-1}(U) \geq \dim U$.
5. If $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of V, and $\vec{w}_1, \ldots, \vec{w}_m$ are any vectors in W, then there is exactly one linear transformation $L : V \to W$ such that $L(\vec{v}_i) = \vec{w}_i$.
6. Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in V. Let $L : \mathbb{R}^n \to V$ be the linear transformation with $L(\vec{e}_i) = \vec{v}_i$. Then
   (a) $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if and only if $\operatorname{Ker} L = \{ \vec{0} \}$.
   (b) $\vec{v}_1, \ldots, \vec{v}_n$ span V if and only if $\operatorname{Range} L = V$.
7. Let $B = \{ \vec{v}_1, \ldots, \vec{v}_n \}$ be a basis of V, $L : V \to W$ be a linear transformation, and $\vec{w}_i = L(\vec{v}_i)$ for $i = 1, \ldots, n$. Then
   (a) $\operatorname{Ker} L = \{ \vec{0} \}$ if and only if $\vec{w}_1, \ldots, \vec{w}_m$ are linearly independent.

(b) $\text{Range}\, L = W$ if and only if $\vec{w}_1, \ldots, \vec{w}_m$ span W.

8. $L : V \to W$ is an isomorphism if and only if $\text{Range}\, L = W$ and $\text{Ker}\, L = \{\, \vec{0}\, \}$.

9. Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of V. Then a linear transformation $L : V \to W$ is an isomorphism if and only if $L(\vec{v_1}), \ldots, L(\vec{v_n})$ is a basis of W.

10. Let $L : V \to W$ be an isomorphism. Recall that this means L is one-to-one and onto, so the inverse function $L^{-1} : W \to V$ exists and is defined by

$$L^{-1}(\vec{w}) = \vec{v} \text{ where } \vec{v} \text{ is the unique vector with } \vec{w} = L(\vec{v}).$$

Show that L^{-1} is a linear transformation. Note: L^{-1} is automatically one-to-one and onto because L is (this is a general fact about inverse functions), so L^{-1} is in fact an isomorphism.

11. What is the domain and target of the function $L \circ L^{-1}$? What is the formula for this function? Same questions for $L^{-1} \circ L$.

12. Let U, V, W be vector spaces. If $K : U \to V$ is a linear transformation and $L : V \to W$ is a linear transformation, then $L \circ K$ is a linear transformation.

13. Let

$$\text{Lin}(V, W) = \{\, L : V \to W \mid L \text{ is a linear transformation}\, \}.$$

Define vector addition and scalar multiplication operations on this set by

$$(L + K)(\vec{v}) = L(\vec{v}) + K(\vec{v}), \quad (\lambda L)(\vec{v}) = \lambda \vec{v}.$$

Then $\text{Lin}(V, W)$ is a vector space with these operations. Remark: An important special case is when $W = \mathbb{R}$, so the vector space is $\text{Lin}(V, \mathbb{R})$. This vector space is called the dual of V and is usually denoted V^*.

14. The vector space $\text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ can be identified with the vector space $M_{m,n}$ of all $m \times n$ matrices. More precisely: There is an obvious linear transformation $\text{Lin}(\mathbb{R}^n, \mathbb{R}^m) \to M_{m,n}$, and it is an isomorphism. What is this obvious linear transformation and why is it an isomorphism?

15. Let A be an $m \times n$ matrix and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $L(\vec{x}) = A\vec{x}$.

    (a) $\text{Range}\, L = \text{col space}\, A$
    (b) $\text{Ker}\, L = \text{null space}\, A$
    (c) $\dim \text{Range}\, L = \text{rank}\, A$
    (d) $\dim \text{Ker}\, L = \text{nullity}\, A$
    (e) $\vec{x}$ is a solution to $A\vec{x} = \vec{0}$ (homogeneous equation) if and only if $\vec{x} \in \text{Ker}\, L$
    (f) the set of solutions to $A\vec{x} = 0$ is $\text{Ker}\, L$

(g) $A\vec{x} = \vec{b}$ (inhomogeneous equation) has a solution if and only if $\vec{b} \in \operatorname{Range} L$

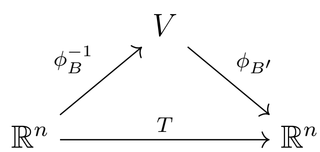
(h) Let $\vec{x_0}$ be a fixed vector with $A\vec{x_0} = \vec{b}$. Then an arbitrary solution to $A\vec{x} = \vec{b}$ can be written as $\vec{x} = \vec{x_0} + \vec{x_h}$ with $\vec{x_h} \in \operatorname{Ker} L$. Geometrically, the set of solutions to $A\vec{x} = \vec{b}$ is the subspace $\operatorname{Ker} L$ shifted by $\vec{x_0}$ (hence an affine space).

(i) L is an isomorphism if and only if A is nonsingular.

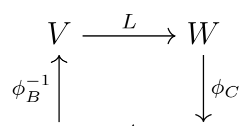
(j) If L is an isomorphism, then $L^{-1}(\vec{x}) = A^{-1}\vec{x}$.

16. Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of V and let $L : V \to \mathbb{R}^n$ be the function $L(\vec{v}) = [\vec{v}]_B$. Then L is an isomorphism.

17. Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $B' = \{\vec{v}_1', \ldots, \vec{v}_n'\}$ be two bases of V. Let $\phi_B : V \to \mathbb{R}^n$ be $\phi_B(\vec{v}) = [\vec{v}]_B$ (i.e. take B coordinates) and $\phi_{B'} : V \to \mathbb{R}^n$ be $\phi_{B'}(\vec{v}) = [\vec{v}]_{B'}$ (i.e. take B' coordinates). Let T be the transformation matrix from B to B' coordinates, so $[\vec{v}]_{B'} = T[\vec{v}]_B$. Then $T\vec{x} = (\phi_{B'} \circ \phi_B^{-1})(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.



18. Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of V, $C = \{\vec{w}_1, \ldots, \vec{w}_m\}$ a basis of W, and $L : V \to W$ a linear transformation. Let A be the matrix representation of L with respect to B and C. Let $\phi_B : V \to \mathbb{R}^n$ be $\phi_B(\vec{v}) = [\vec{v}]_B$ and let $\phi_C : W \to \mathbb{R}^m$ be $\phi_C(\vec{w}) = [\vec{w}]_C$. Then $A\vec{x} = (\phi_C \circ L \circ \phi_B^{-1})(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.



19. Continuing the notation of the previous problem,

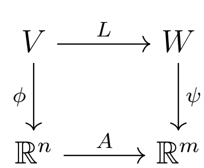
(a) $\phi_B(\operatorname{Ker} L) = \text{null space } A$

(b) $\phi_C(\operatorname{Range} L) = \text{col space } A$

(c) $\dim \text{null space } A + \dim \text{col space } A = \dim \operatorname{Ker} L + \dim \operatorname{Range} L = \dim V = n$

20. If $L : V \to \mathbb{R}^n$ is an isomorphism, then there exists a basis B of V for which $L(\vec{v}) = [\vec{v}]_B$.

21. A linear transformation $L : V \to W$ can be represented by a matrix by choosing isomorphisms $\phi : V \to \mathbb{R}^n$, $\psi : W \to \mathbb{R}^m$, and taking the matrix to be the matrix A which fills in the bottom arrow of the diagram



Why is this compatible with the usual definition of the matrix representation of a linear transformation?