Linear Algebra, Spring 2016
Homework 8, Due Tuesday, April 12

Problem 1 (3 points) We defined $\mathrm{Det}(A)$ for A a square matrix. The goal of this problem is to define $\mathrm{Det}(f)$ for $f: V \to V$ a linear transformation (and V an arbitrary finite dimensional vector space). To do this, we begin as follows: Let B be any basis of V, and let A_f be the matrix for f with respect to the basis B. Then we want to define $\mathrm{Det}(f)$ to be $\mathrm{Det}(A_f)$. However, for this to be a good definition, we need to check that it does not depend on the choice of basis. Check this. That is, if B' is another basis and A'_f is the matrix for f with respect to B', show that $\mathrm{Det}(A_f) = \mathrm{Det}(A'_f)$.

We know
$$\begin{cases} A_f[\vec{v}]_B = [f(\vec{v})]_B \\ A_f'[\vec{v}]_{B'} = [f(\vec{v})]_{B'} \\ T[\vec{v}]_B = [\vec{v}]_{B'} \quad \left(\text{for some invertible matrix } T\right) \end{cases}$$

So:
$$A_f'[\vec{v}]_{B'} = A_f' T [\vec{v}]_B$$

and also
$$A_f'[\vec{v}]_{B'} = [f(\vec{v})]_{B'} = T[f(\vec{v})]_B = TA_f[\vec{v}]_B.$$

So $A_f' T[\vec{v}]_B = TA_f[\vec{v}]_B$ for all $\vec{v}$, so

$$A_f' T = TA_f.$$

Thus
$$\mathrm{Det}(A_f')\,\mathrm{Det}(T) = \mathrm{Det}(T)\,\mathrm{Det}(A_f)$$

So
$$\mathrm{Det}(A_f') = \mathrm{Det}(A_f) \quad \left(\text{since Det} \neq 0\right)$$

Problem 2 (3 points) In single variable calculus, a function $f : \mathbb{R} \to \mathbb{R}$ can be approximated by its tangent line. This type of approximation is known as a linearization. In this problem, we investigate the 2-dimensional analog of linearization. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. We can write f as

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix},$$

for some functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. The *derivative* of f at a point (x,y) is the linear transformation $Df_{(x,y)}\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$Df_{(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix}.$$

The *linearization* of f at the point (x_0, y_0) is the function $Lf_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$Lf_{(x_0,y_0)}(x,y) = \underbrace{f(x_0,y_0)}_{\text{2-vector}} + \underbrace{Df_{(x_0,y_0)}}_{2\times 2 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{\text{2-vector}}.$$

$$\begin{bmatrix} h - 0 \\ 2h - 0 \end{bmatrix}$$

Now let

$$f(x,y) = \begin{bmatrix} \sqrt{1 + 3x + y} \\ \cos(x) + 2\sin(y) \end{bmatrix}.$$

Find $Lf_{(0,0)}$. Use your answer to find a first order approximation of $f(h, 2h)$ for h small. (First order approximation means linearization, so you just need to evaluate your answer at $(h, 2h)$.)

$$Df_{(x,y)} = \begin{bmatrix} \frac{3}{2\sqrt{1+3x+y}} & \frac{1}{\sqrt{1+3x+y}} \\ -\sin x & 2\cos y \end{bmatrix}. \quad Df_{(0,0)} = \begin{bmatrix} 3/2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$f(0,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus $$f(h,2h) \approx Lf_{(0,0)}(h,2h) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3/2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} h \\ 2h \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 7/2\, h \\ 1 + 4h \end{bmatrix}$$

Problem 3 (3 points) This problem shows how to use determinants to find the area of curved regions. Fix a constant $R > 0$. Let $[0, R] \times [0, 2\pi]$ be the rectangle with coordinates (r, θ) with $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Consider the function $f : [0, R] \times [0, 2\pi] \to \mathbb{R}^2$ defined by

$$f(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}.$$

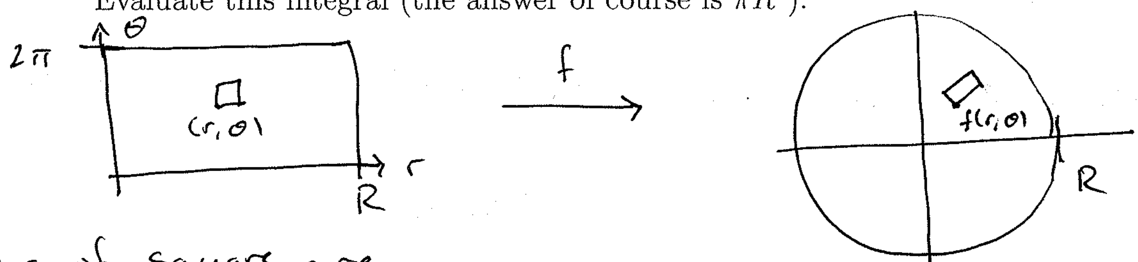
The image of f is a circle of radius R. Now imagine what happens under the mapping f to a little square with sides of length h and a corner at (r, θ) (see the picture). Since h is small, f can be approximated at (r, θ) by $Lf_{(r,\theta)}$, and since this is a map of the form (linear map)+(constant vector), it maps the little square to a little parallelogram. The square has sides of length h, and the linear map part of the linearization is $Df_{(r,\theta)}$, so the area of the parallelogram is $h^2|\operatorname{Det} Df_{(r,\theta)}|$ (we use the absolute value so we don't have to worry about $\pm$-orientation issues). Since the circle can be approximately covered by the parallelograms, we have

$$\text{Area(circle)} \approx \sum_{\text{little squares}} |\operatorname{Det} Df_{(r,\theta)}| h^2.$$

Letting $h \to 0$ makes the approximation become exact. Since h^2 is the area of a little square, as $h \to 0$, the right-hand side goes to the double integral of $|\operatorname{Det} Df_{(r,\theta)}|$ over the region $[0, R] \times [0, 2\pi]$. Thus

$$\text{Area(circle)} = \int_0^{2\pi}\int_0^R |\operatorname{Det} Df_{(r,\theta)}| \, dr\, d\theta.$$

Evaluate this integral (the answer of course is πR^2).



sides of square are $h\vec{e}_1$, $h\vec{e}_2$

sides of parallelogram are $h\,Df_{(r,\theta)}\cdot\vec{e}_1$, $h\,Df_{(r,\theta)}\cdot\vec{e}_2$

$$Df = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}, \quad |\operatorname{Det} Df| = |r\cos^2\theta + r\sin^2\theta| = r$$

So $\text{Area(circle)} = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = 2\pi \cdot \frac{r^2}{2}\Big|_0^R = \pi R^2$

Problem 4 (3 points) The upshot of the previous problem is that for a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{Det}(Df)$ measures the infinitesimal distortion of n-dimensional volume under the mapping f. Here is a 3-dimensional example involving the solid sphere, with the function f based on spherical coordinates from multivariable calculus. Consider the 3d box $[0, R] \times [0, 2\pi] \times [0, \pi]$ with coordinates (r, θ, ϕ). Let $f : [0, R] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ be the function

$$f(r, \theta, \phi) = \begin{bmatrix} f_1(r,\theta,\phi) \\ f_2(r,\theta,\phi) \\ f_3(r,\theta,\phi) \end{bmatrix} = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}.$$

The image of f is a solid sphere of radius R. The derivative of f at (r, θ, ϕ) is the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$Df_{(r,\theta,\phi)} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \phi} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \phi} \end{bmatrix}$$

Calculate $|\,\mathrm{Det}\, Df_{(r,\theta,\phi)}|$.

$$\mathrm{Det}\, Df = \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & +r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix} = \cos\varphi \begin{vmatrix} -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}$$

$$-r\sin\varphi \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi \end{vmatrix} = \cos\varphi\left[-r^2\sin^2\theta\sin\varphi\cos\varphi - r^2\cos^2\theta\sin\varphi\cos\varphi\right]$$

$$-r\sin\varphi\left[r\cos^2\theta\sin^2\varphi + r\sin^2\theta\sin^2\varphi\right] = -r^2\cos^2\varphi\sin\varphi - r^2\sin^2\varphi\sin\varphi$$

$$= -r^2\sin\varphi$$

So $|\mathrm{Det}\, Df| = r^2\sin\varphi$ (always positive b/c $r > 0$, $0 \le \varphi \le \pi$)

Problem 5 (3 points) Evaluate the triple integral

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} |\operatorname{Det} Df_{(r,\theta,\phi)}| dr d\theta d\phi.$$

(Similarly to Problem 3, this integral gives the volume of the solid sphere of radius R, so the answer is $4\pi R^3/3$.)

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\varphi \, d\varphi \, d\theta \, dr = 4\pi \int_0^R r^2 \, dr = \frac{4\pi R^3}{3}$$