Linear Algebra, Spring 2016
Homework 8, Due Tuesday, April 12

Problem 1 (3 points) We defined $\mathrm{Det}(A)$ for A a square matrix. The goal of this problem is to define $\mathrm{Det}(f)$ for $f : V \to V$ a linear transformation (and V an arbitrary finite dimensional vector space). To do this, we begin as follows: Let B be any basis of V, and let A_f be the matrix for f with respect to the basis B. Then we want to define $\mathrm{Det}(f)$ to be $\mathrm{Det}(A_f)$. However, for this to be a good definition, we need to check that it does not depend on the choice of basis. Check this. That is, if B' is another basis and A'_f is the matrix for f with respect to B', show that $\mathrm{Det}(A_f) = \mathrm{Det}(A'_f)$.

Problem 2 (3 points) In single variable calculus, a function $f : \mathbb{R} \to \mathbb{R}$ can be approximated by its tangent line. This type of approximation is known as a linearization. In this problem, we investigate the 2-dimensional analog of linearization. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. We can write f as

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix},$$

for some functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. The *derivative* of f at a point (x, y) is the linear transformation $Df_{(x,y)}\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$Df_{(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x, y) & \frac{\partial f_1}{\partial y}(x, y) \\ \frac{\partial f_2}{\partial x}(x, y) & \frac{\partial f_2}{\partial y}(x, y) \end{bmatrix}.$$

The *linearization* of f at the point (x_0, y_0) is the function $Lf_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$Lf_{(x_0,y_0)}(x, y) = f(x_0, y_0) + Df_{(x_0,y_0)} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

Now let

$$f(x, y) = \begin{bmatrix} \sqrt{1 + 3x + y} \\ \cos(x) + 2\sin(y) \end{bmatrix}.$$

Find $Lf_{(0,0)}$. Use your answer to find a first order approximation of $f(h, 2h)$ for h small. (First order approximation means linearization, so you just need to evaluate your answer at $(h, 2h)$.)

Problem 3 (3 points) This problem shows how to use determinants to find the area of curved regions. Fix a constant $R > 0$. Let $[0, R] \times [0, 2\pi]$ be the rectangle with coordinates (r, θ) with $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Consider the function $f : [0, R] \times [0, 2\pi] \to \mathbb{R}^2$ defined by

$$f(r, \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}.$$

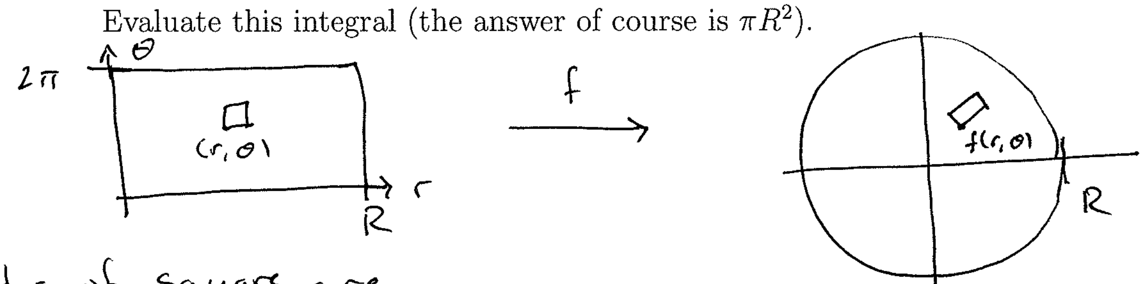
The image of f is a circle of radius R. Now imagine what happens under the mapping f to a little square with sides of length h and a corner at (r, θ) (see the picture). Since h is small, f can be approximated at (r, θ) by $Lf_{(r,\theta)}$, and since this is a map of the form (linear map)+(constant vector), it maps the little square to a little parallelogram. The square has sides of length h, and the linear map part of the linearization is $Df_{(r,\theta)}$, so the area of the parallelogram is $h^2|\operatorname{Det} Df_{(r,\theta)}|$ (we use the absolute value so we don't have to worry about $\pm$-orientation issues). Since the circle can be approximately covered by the parallelograms, we have

$$\text{Area(circle)} \approx \sum_{\text{little squares}} |\operatorname{Det} Df_{(r,\theta)}| h^2.$$

Letting $h \to 0$ makes the approximation become exact. Since h^2 is the area of a little square, as $h \to 0$, the right-hand side goes to the double integral of $|\operatorname{Det} Df_{(r,\theta)}|$ over the region $[0, R] \times [0, 2\pi]$. Thus

$$\text{Area(circle)} = \int_0^{2\pi} \int_0^R |\operatorname{Det} Df_{(r,\theta)}| \, dr \, d\theta.$$

Evaluate this integral (the answer of course is πR^2).



sides of square are $h\vec{e_1}$, $h\vec{e_2}$

sides of parallelogram are $h\, Df_{(r,\theta)} \cdot \vec{e_1}$, $h\, Df_{(r,\theta)} \cdot \vec{e_2}$

Problem 4 (3 points) The upshot of the previous problem is that for a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\text{Det}(Df)$ measures the infinitesimal distortion of n-dimensional volume under the mapping f. Here is a 3-dimensional example involving the solid sphere, with the function f based on spherical coordinates from multivariable calculus. Consider the 3d box $[0, R] \times [0, 2\pi] \times [0, \pi]$ with coordinates (r, θ, ϕ). Let $f : [0, R] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ be the function

$$f(r,\theta,\phi) = \begin{bmatrix} f_1(r,\theta,\phi) \\ f_2(r,\theta,\phi) \\ f_3(r,\theta,\phi) \end{bmatrix} = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}.$$

The image of f is a solid sphere of radius R. The derivative of f at (r, θ, ϕ) is the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$Df_{(r,\theta,\phi)} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \phi} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \phi} \end{bmatrix}$$

Calculate $|\,\text{Det}\, Df_{(r,\theta,\phi)}|$.

Problem 5 (3 points) Evaluate the triple integral

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} |\operatorname{Det} Df_{(r,\theta,\phi)}| dr d\theta d\phi.$$

(Similarly to Problem 3, this integral gives the volume of the solid sphere of radius R, so the answer is $4\pi R^3/3$.)