Linear Algebra, Spring 2016
Homework 6, Due Thursday, March 24

The goal of this homework is to find a formula for the n^{th} term of the Fibonacci sequence. Recall (from calculus) that a *sequence* is a list of numbers. Equivalently, a sequence can be thought of as a function $f : \mathbb{N} \to \mathbb{R}$ (remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$): the function f corresponds to the list $f(1), f(2), f(3), \ldots$. Sequences can be added and multiplied by scalars. Using function notation, the sum of the sequences f and g is the sequence $f + g$ defined by $(f + g)(n) = f(n) + g(n)$, and for $c \in \mathbb{R}$ the sequence cf is defined by $(cf)(n) = cf(n)$. With these operations, the set of all sequences is a vector space. Call this vector sapce V.

The *Fibonacci sequence* is the sequence

$$1, 1, 2, 3, 5, 8, \ldots.$$

It has the property that the first two terms are 1, and after that each term is the sum of the previous two terms. In other words, if we let $F_{1,1} : \mathbb{N} \to \mathbb{R}$ denote this sequence, then

$$F_{1,1}(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2, \\ F_{1,1}(n-1) + F_{1,1}(n-2) & \text{if } n \geq 3. \end{cases}$$

This type of definition is called a *recursive definition*. It has the drawback that in order to find say the 1000^{th} term, the first 999 terms need to be found first. In this homework assignment you will find a non-recursive formula for $F_{1,1}(n)$ (this is sometimes called a *closed formula*).

For any real numbers a and b, the *generalized Fibonacci sequence* starting with a, b is the sequence

$$a, b, a + b, a + 2b, 2a + 3b, \ldots.$$

It is defined in the same way as the usual Fibonacci sequence except that the first two terms are a, b instead of $1, 1$. We denote this sequence as $F_{a,b}$. Since $F_{a,b}$ is a sequence, it is an element of the vector space V of all sequences. Let

$$W = \{F_{a,b} \mid a, b \in \mathbb{R}\}.$$

Notice that W is a subset of V.

Problem 1 (3 points) Show that W is a *subspace* of V. (To do this, you need to show that W is closed under vector addition and scalar multiplication. Hint: Show that $cF_{a,b} = F_{ca,cb}$ and $F_{a,b} + F_{c,d} = F_{a+c,b+d}$. You will probably need to use induction to show the latter equality.)

See next pages

Problem 2 (3 points) Show that the dimension of W is 2.

From the hint in Problem 1:

$F_{a,b} = a F_{1,0} + b F_{0,1}$, so $\{F_{1,0}, F_{0,1}\}$ spans W

Also, if $x F_{1,0} + y F_{0,1} = F_{0,0}$ ($= \vec{0}$ in W)

then $F_{x,y} = F_{0,0}$. But then

$x = F_{x,y}(1) = F_{0,0}(1) = 0$

$y = F_{x,y}(2) = F_{0,0}(2) = 0$.

So $\{F_{1,0}, F_{0,1}\}$ is a lin. ind. set.

Hence it is a basis for W and $\dim W = 2$

closed under scalar multiplication. :

Given $F_{a,b} \in W$ and $\lambda \in \mathbb{R}$ we need to show that $\lambda F_{a,b} \in W$.

To show that $\lambda F_{a,b} \in W$, we need to show that $\lambda F_{a,b}$ is a Fibonacci sequence. Since $F_{\lambda a, \lambda b}$ is a Fibonacci sequence, it suffices to show that $\lambda F_{a,b} = F_{\lambda a, \lambda b}$.

This equation is equivalent to $\lambda F_{a,b}(n) = F_{\lambda a, \lambda b}(n)$ for all n. (*)

We prove (*) by induction:

For $n=1$:

$$\lambda F_{a,b}(1) = \lambda a$$

$$F_{\lambda a, \lambda b}(1) = \lambda a$$

For $n=2$:

$$\lambda F_{a,b}(2) = \lambda b$$

$$F_{\lambda a, \lambda b}(2) = \lambda b$$

Assume true for $n \leq N$, prove for $n = N+1$:

$$\lambda F_{a,b}(N+1) = \lambda \left[F_{a,b}(N) + F_{a,b}(N-1) \right] \quad \left(\text{by dfn. of Fib. seq.}\right)$$

$$= \lambda F_{a,b}(N) + \lambda F_{a,b}(N-1)$$

$$= F_{\lambda a, \lambda b}(N) + F_{\lambda a, \lambda b}(N-1) \quad \left(\text{by inductive assumption}\right)$$

$$= F_{\lambda a, \lambda b}(N+1) \quad \left(\text{by dfn. of Fib. seq.}\right)$$

Thus (*) is true for all n by induction, and so

$\lambda F_{a,b} = F_{\lambda a, \lambda b} \in W$, i.e. W is closed under scalar mult.

Closed under vecl

Closed under vector addition:

Given $F_{a,b}$ and $F_{c,d}$ in W we need to show that $F_{a,b} + F_{c,d}$ is in W. It suffices to show that $F_{a,b} + F_{c,d} = F_{a+c,\,b+d}$,

i.e. (*) $\quad F_{a,b}(n) + F_{c,d}(n) = F_{a+c,\,b+d}(n)$ for all n.

We prove (*) by induction:

For $n=1$: $\quad F_{a,b}(1) + F_{c,d}(1) = a + c$

$F_{a+c,\,b+d}(1) = a + c$

For $n=2$: $\quad F_{a,b}(2) + F_{c,d}(2) = b + d$

$F_{a+c,\,b+d}(2) = b + d$

Assume true for $n \le N$, prove for $n = N+1$:

$$
\begin{aligned}
F_{a,b}(N+1) + F_{c,d}(N+1) &= F_{a,b}(N) + F_{a,b}(N-1) && \left(\text{by dfn. of Fib. seq.}\right)\\
&\quad + F_{c,d}(N) + F_{c,d}(N-1)\\
&= \left[F_{a,b}(N) + F_{c,d}(N)\right] + \left[F_{a,b}(N-1) + F_{c,d}(N-1)\right]\\
&= F_{a+c,\,b+d}(N) + F_{a+c,\,b+d}(N-1) && \left(\text{by inductive assumption}\right)\\
&= F_{a+c,\,b+d}(N+1) && \left(\text{by dfn. of Fib. seq.}\right)
\end{aligned}
$$

Thus $F_{a,b} + F_{c,d} = F_{a+c,\,b+d} \in W$, so W is closed under vector addition

Problem 3 (3 points) Let x be a real number satisfying $x+1=x^2$ (this in fact means that $x=(1\pm\sqrt{5})/2$, but you don't need this fact to do this problem). Show that $F_{1,x}$ is the sequence

$$1, x, x^2, x^3, x^4, \ldots.$$

Notice $x^n + x^{n-1} = x^{n-1}(x+1) = x^{n-1}\cdot x^2 = x^{n+1}$ for all n.

Thus $F_{1,x}$ is the sequence

$1, x, 1+x=x^2, x+x^2=x^3, x^2+x^3=x^4, \ldots$

i.e. $F_{1,x}(n) = x^{n-1}$ when $x = \dfrac{1\pm\sqrt{5}}{2}$

Problem 4 (3 points) Let $\lambda=(1+\sqrt{5})/2$ and $\sigma=(1-\sqrt{5}/2)$. Show that $\{F_{1,\lambda}, F_{1,\sigma}\}$ is a basis of W, and find scalars x and y such that

$$F_{1,1} = xF_{1,\lambda} + yF_{1,\sigma}.$$

Suppose $x_1 F_{1,\lambda} + x_2 F_{1,\sigma} = F_{0,0} \; (= \vec{0} \text{ in } V)$

Then: $\begin{cases} x_1 F_{1,\lambda}(1) + x_2 F_{1,\sigma}(1) = 0 \\ x_1 F_{1,\lambda}(2) + x_2 F_{1,\sigma}(2) = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1\lambda + x_2\sigma = 0 \end{cases} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$

So $F_{1,\lambda}, F_{1,\sigma}$ are lin. ind. Since $\dim W = 2$, they form a basis.

To solve $F_{1,1} = xF_{1,\lambda} + yF_{1,\sigma}$, need to solve (using hint to problem 1)

$F_{1,1} = F_{x+y,\, x\lambda + y\sigma}$. I.e. solve $\begin{cases} x+y=1 \\ x\lambda + y\sigma = 1 \end{cases}$

Solution is $\begin{matrix} x = \lambda/\sqrt{5} \\ y = -\sigma/\sqrt{5} \end{matrix}$, so $\boxed{F_{1,1} = \dfrac{\lambda}{\sqrt{5}} F_{1,\lambda} - \dfrac{\sigma}{\sqrt{5}} F_{1,\sigma}}$

Problem 5 (3 points) Find a non-recursive formula for $F_{1,1}(n)$.

From Problem 4) and Problem 3):

$$F_{1,1}(n) = \frac{\lambda}{\sqrt{5}} F_{1,\lambda}(n) - \frac{\sigma}{\sqrt{5}} F_{1,\sigma}(n) = \frac{\lambda}{\sqrt{5}} \cdot \lambda^{n-1} - \frac{\sigma}{\sqrt{5}} \cdot \sigma^{n-1}.$$

So

$$\boxed{F_{1,1}(n) = \frac{\lambda^n}{\sqrt{5}} - \frac{\sigma^n}{\sqrt{5}}}$$

Test it!

$$F_{1,1}(1000) \approx 4.3467 \times 10^{208}$$