Linear Algebra, Spring 2016
Homework 6, Due Thursday, March 24

The goal of this homework is to find a formula for the n^{th} term of the Fibonacci sequence. Recall (from calculus) that a *sequence* is a list of numbers. Equivalently, a sequence can be thought of as a function $f : \mathbb{N} \to \mathbb{R}$ (remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$): the function f corresponds to the list $f(1), f(2), f(3), \ldots$. Sequences can be added and multiplied by scalars. Using function notation, the sum of the sequences f and g is the sequence $f + g$ defined by $(f + g)(n) = f(n) + g(n)$, and for $c \in \mathbb{R}$ the sequence cf is defined by $(cf)(n) = cf(n)$. With these operations, the set of all sequences is a vector space. Call this vector sapce V.

The *Fibonacci sequence* is the sequence

$$1, 1, 2, 3, 5, 8, \ldots .$$

It has the property that the first two terms are 1, and after that each term is the sum of the previous two terms. In other words, if we let $F_{1,1} : \mathbb{N} \to \mathbb{R}$ denote this sequence, then

$$F_{1,1}(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2, \\ F_{1,1}(n-1) + F_{1,1}(n-2) & \text{if } n \geq 3. \end{cases}$$

This type of definition is called a *recursive definition*. It has the drawback that in order to find say the 1000^{th} term, the first 999 terms need to be found first. In this homework assignment you will find a non-recursive formula for $F_{1,1}(n)$ (this is sometimes called a *closed formula*).

For any real numbers a and b, the *generalized Fibonacci sequence* starting with a, b is the sequence

$$a, b, a + b, a + 2b, 2a + 3b, \ldots .$$

It is defined in the same way as the usual Fibonacci sequence except that the first two terms are a, b instead of $1, 1$. We denote this sequence as $F_{a,b}$. Since $F_{a,b}$ is a sequence, it is an element of the vector space V of all sequences. Let

$$W = \{F_{a,b} \mid a, b \in \mathbb{R}\}.$$

Notice that W is a subset of V.

Problem 1 (3 points) Show that W is a *subspace* of V. (To do this, you need to show that W is closed under vector addition and scalar multiplication. Hint: Show that $cF_{a,b} = F_{ca,cb}$ and $F_{a,b} + F_{c,d} = F_{a+c,b+d}$. You will probably need to use induction to show the latter equality.)

Problem 2 (3 points) Show that the dimension of W is 2.

Problem 3 (3 points) Let x be a real number satisfying $x+1=x^2$ (this in fact means that $x=(1\pm\sqrt{5})/2$, but you don't need this fact to do this problem). Show that $F_{1,x}$ is the sequence

$$1, x, x^2, x^3, x^4, \ldots.$$

Problem 4 (3 points) Let $\lambda=(1+\sqrt{5})/2$ and $\sigma=(1-\sqrt{5}/2)$. Show that $\{F_{1,\lambda}, F_{1,\sigma}\}$ is a basis of W, and find scalars x and y such that

$$F_{1,1}=xF_{1,\lambda}+yF_{1,\sigma}.$$

Problem 5 (3 points) Find a non-recursive formula for $F_{1,1}(n)$.