Linear Algebra, Spring 2016
Homework 3

Name: SOLUTIONS

Problem 1: (3 points) Recall that the **dot product** of two vectors in $\mathbb{R}^3$ is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz,$$

and this is essentially the same as the matrix multiplication $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Two vectors $\vec{u}$ and $\vec{v}$ are **perpendicular** if and only if $\vec{u} \cdot \vec{v} = 0$. The **length** of a vector $\vec{u}$ is $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$. A vector $\vec{u}$ is a **unit vector** if $|\vec{u}| = 1$. If $\vec{u} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ then the **cross product** of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = \begin{bmatrix} bz - cy \\ -az + cx \\ ay - bx \end{bmatrix}.$$

The cross product $\vec{u} \times \vec{v}$ has the property that it is perpendicular to $\vec{u}$ and $\vec{v}$, it is a unit vector if $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, and the vectors $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ conform to the right-hand rule (more on this in problem 2).

a) Let $\vec{u}' = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$. Find a unit vector $\vec{u}$ that has the same direction as $\vec{u}'$.

$$\vec{u}' = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad |\vec{u}'| = \sqrt{9} = 3, \quad \vec{u} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

b) Find a unit vector $\vec{v}$ that is perpendicular to $\vec{u}$.

$$\vec{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

c) Find a unit vector $\vec{w}$ that is perpendicular to $\vec{u}$ and $\vec{v}$.

$$\vec{W} = \vec{u} \times \vec{v} = \begin{bmatrix} 2/3 \cdot 0 - 1/3 \cdot -1/\sqrt{2} \\ -2/3 \cdot 0 + 1/3 \cdot 1/\sqrt{2} \\ 2/3 \cdot -1/\sqrt{2} - 2/3 \cdot 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}$$

Problem 2: (3 points) Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be a list of three perpendicular vectors in $\mathbb{R}^3$. We say they satisfy the **right-hand rule** if, using your right hand, when you point your index finger in the direction of $\vec{v_1}$ and your middle finger in the direction of $\vec{v_2}$, your thumb then points in the direction of $\vec{v_3}$. The standard basis vectors in the order $\vec{e_1}, \vec{e_2}, \vec{e_3}$ satisfy the right-hand rule, whereas in the order $\vec{e_2}, \vec{e_1}, \vec{e_3}$ they do not (in this order they satisfy the left-hand rule). To check if the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ satisfy the right-hand rule, form the matrix $A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}$. Then the **determinant** of the matrix is positive if and only if the vectors satisfy the right-hand rule. We will learn the formula for a determinant later in the class, for now you can use the command det in MatLab or your calculator to calculate it.

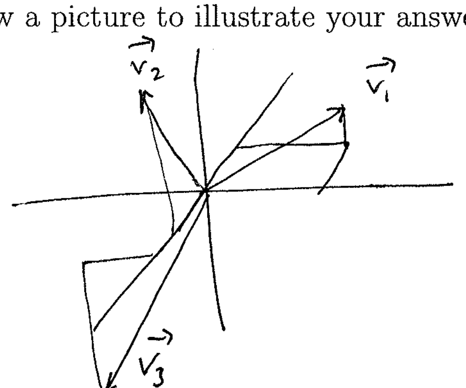
a) Let

$$\vec{v_1} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix}.$$

Does $\vec{v_1}, \vec{v_2}, \vec{v_3}$ satisfy the right-hand rule?

$\det \begin{bmatrix} -2 & 1 & 3 \\ 3 & 0 & -2 \\ 1 & 4 & -7 \end{bmatrix} = 39 > 0$, yes - satisfies right hand rule

b) Try to draw a picture to illustrate your answer from part a).



c) Does $-\vec{v_1}, \vec{v_2}, \vec{v_3}$ satisfy the right-hand rule?

No, $\vec{v_1}, \vec{v_2}, \vec{v_3}$ right-handed

$\Rightarrow$ $-\vec{v_1}, \vec{v_2}, \vec{v_3}$ left-handed

Problem 3: (3 points) A **rotation of 3-space** is a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ that takes the standard basis vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$ to three perpendicular vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ which satisfy the right-hand rule. Take a moment to think about this definition and convince yourself it is reasonable. For instance, convince yourself that rotations of 2-space have a similar property. Since $L(\vec{e_1}) = \vec{v_1}, L(\vec{e_2}) = \vec{v_2}, L(\vec{e_3}) = \vec{v_3}$, the matrix of the linear transformation L is

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}.$$

For example, the rotation of 3-space that takes the x-axis to the y-axis, the y-axis to the z-axis, and the z-axis to the x-axis has the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The following problems can be answered by figuring out what the rotation does to the standard basis vectors. Your knowledge of rotations in 2-space will be needed.

a) Find the matrix that rotates the xy-plane by $\pi/2$ and leaves the z-axis fixed.

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Find the matrix that rotates the yz-plane by $-\pi/3$ and leaves the x-axis fixed.

rotation by $-\pi/3$ in plane $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/2 & +\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

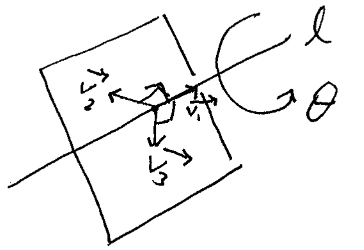
$\Rightarrow$ rotation of 3-space is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$

c) Find the matrix that is the following composition of rotations: first rotate by the matrix in part a), then by the matrix in part b).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

Problem 4: (3 points) Given a line ℓ and an angle θ, let L be the rotation that leaves ℓ fixed and rotates the plane perpendicular to ℓ by an angle of θ. The goal of this problem is to find the matrix for L. To do this, we will first write down how L acts on a set of vectors which are conveniently adapted to the the setup of the problem, and then deduce what L is from this information. So to begin with, let $\vec{v_1}$ be a unit vector in the direction of ℓ, and let $\vec{v_2}, \vec{v_3}$ be two other unit vectors so that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are all perpendicular and satisfy the right-hand rule. This will be our convenient set of vectors (you can think of this as a "change of coordinates" or "change of basis").

a) Draw a picture that illustrates the setup.



b) Fill in the blanks for how L acts on the vectors. Knowledge of rotations in 2-space will be needed!

$$L(\vec{v_1}) = \underline{\quad 1 \quad}\vec{v_1}$$

$$L(\vec{v_2}) = \underline{\quad \cos\theta \quad}\vec{v_2} + \underline{\quad \sin\theta \quad}\vec{v_3}$$

$$L(\vec{v_3}) = \underline{\quad -\sin\theta \quad}\vec{v_2} + \underline{\quad \cos\theta \quad}\vec{v_3}.$$

Thinking of L as a matrix, this means that

$$L\begin{bmatrix}\vec{v_1} & \vec{v_2} & \vec{v_3}\end{bmatrix} = \begin{bmatrix}\vec{v_1} & \cos\theta\,\vec{v_2} + \sin\theta\,\vec{v_3} & -\sin\theta\,\vec{v_2} + \cos\theta\,\vec{v_3}\end{bmatrix}.$$

Using matrix inverses, the matrix L can be written as

$$L = \begin{bmatrix}\vec{v_1} & \cos\theta\,\vec{v_2} + \sin\theta\,\vec{v_3} & -\sin\theta\,\vec{v_2} + \cos\theta\,\vec{v_3}\end{bmatrix}\begin{bmatrix}\vec{v_1} & \vec{v_2} & \vec{v_3}\end{bmatrix}^{-1}$$

c) Find the matrix for L when the line ℓ is in the direction of $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ and $\theta = \pi/4$. Same question for $\theta = 2\pi/3$. Convince yourself geometrically that your answer for $\theta = 2\pi/3$ makes sense by thinking about what happens to the x-, y-, and z-axes under the rotation.

$$\vec{v_1} = \begin{bmatrix}1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3}\end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix}1/\sqrt{2} \\ -1/\sqrt{2} \\ 0\end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix}1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6}\end{bmatrix}$$

$$\begin{bmatrix}.81 & -.31 & .51 \\ .51 & .81 & -.31 \\ -.31 & .51 & .81\end{bmatrix}$$

For $\theta = \pi/4$: $L = \begin{bmatrix}\vec{v_1} & \frac{1}{\sqrt{2}}\vec{v_2} + \frac{1}{\sqrt{2}}\vec{v_3} & -\frac{1}{\sqrt{2}}\vec{v_2} + \frac{1}{\sqrt{2}}\vec{v_3}\end{bmatrix}\begin{bmatrix}\vec{v_1} & \vec{v_2} & \vec{v_3}\end{bmatrix}^{-1} = \cancel{\begin{bmatrix}.84 & -.31 & .47 \\ .51 & .77 & -.28 \\ -.35 & .55 & .80\end{bmatrix}}$

For $\theta = \frac{2\pi}{3}$: $L = \begin{bmatrix}\vec{v_1} & -\frac{1}{2}\vec{v_2} + \frac{\sqrt{3}}{2}\vec{v_3} & -\frac{\sqrt{3}}{2}\vec{v_2} + \frac{-1}{2}\vec{v_3}\end{bmatrix}\begin{bmatrix}\vec{v_1} & \vec{v_2} & \vec{v_3}\end{bmatrix}^{-1} = \begin{bmatrix}0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}$

Problem 5: (3 points)
a) Let A be a rotation matrix. Prove that $A^T A = I_3$. (It actually is also true that $AA^T = I_3$.) Hint: Think of matrix multiplication in terms of (row)(column), i.e. dot products, and use the fact that the columns of A are perpendicular unit vectors.

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3], \quad \left.\begin{matrix}\vec{v}_1\cdot\vec{v}_1 = 1\\ \vec{v}_2\cdot\vec{v}_2 = 1\\ \vec{v}_3\cdot\vec{v}_3 = 1\end{matrix}\right] \vec{v}_i\cdot\vec{v}_j = 1 \text{ if } i=j \quad \left.\begin{matrix}\vec{v}_1\cdot\vec{v}_2 = 0\\ \vec{v}_2\cdot\vec{v}_3 = 0\\ \vec{v}_1\cdot\vec{v}_3 = 0\end{matrix}\right] \vec{v}_i\cdot\vec{v}_j = 0 \text{ if } i\neq j$$

$$A^T A = \begin{bmatrix}\vec{v}_1^{\,T}\\ \vec{v}_2^{\,T}\\ \vec{v}_3^{\,T}\end{bmatrix}[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = [\underbrace{\vec{v}_i\cdot\vec{v}_j}_{i,j \text{ entry}}] = I_3$$

b) Let $A(t)$ be a rotation matrix that depends on time t, with $A(0) = I_3$. This simply means that the entries of the matrix depend on time t, and for each instant t, the matrix is a rotation matrix. $A(t)$ can, for instance, be thought of as giving the orientation of a rigid body spinning in 3-space as a function of time. The matrix can be differentiated with respect to time simply by differentiating each of its entries (this is similar to how time dependent vectors are differentiated in calculus). Prove that $A'(0)^T = -A'(0)$. That is, the derivative at time 0 is a skew symmetric matrix. Hint: Use the product rule to implicitly differentiate the equation $A(t)^T A(t) = I_3$ and then plug in $t = 0$.

$$A(t)^T A(t) = I_3 \implies \frac{d}{dt}\left[A(t)^T A(t)\right] = \frac{d}{dt}(I_3)$$

$$\underset{\text{(product rule)}}{\implies} \left(\frac{d}{dt}A(t)^T\right)A(t) + A(t)^T\left(\frac{d}{dt}A(t)\right) = 0 \underset{\text{plug in } t=0}{\implies} A'(0)^T A(0) + A(0)^T A'(0) = 0$$

$$\implies A'(0)^T I_3 + I_3^T A'(0) = 0 \implies A'(0)^T + A'(0) = 0 \implies A'(0)^T = -A'(0)$$

c) The derivative $A'(t)$ can be thought of as a 3-space version of "angular velocity". Imagine taking a rigid body at rest, and then giving it some initial angular velocity. How many degrees of freedom are there in the set of all possible initial angular velocities? (Hint: How many degrees of freedom are there in the set of all 3x3 skew symmetric matrices?) This number can be thought of as the number of degrees of freedom in the set of all rotations; in other words, the number of independent directions a rigid body could be rotated in at any instant.

arbitrary skew-symmetric matrix is of form $\begin{bmatrix}0 & -s & -t\\ s & 0 & -u\\ t & u & 0\end{bmatrix}$

3 parameters $\implies$ $\boxed{\text{3 degrees of freedom}}$

Remark: It is an accident that the # of degrees of freedom equals the 3 in $\mathbb{R}^3$ ... for $\mathbb{R}^n$ with $n \neq 3$ this doesn't happen!
Think of $\mathbb{R}^2$ for example...