True/False

1. Which of the following sets are subspaces of $\mathbb{R}^3$? Circle yes if it is a subspace and no if it is not. (15 pts)

   (a) A line in $\mathbb{R}^3$ which does not go through the origin. yes/no

   (b) A plane through the origin in $\mathbb{R}^3$. yes/no

   (c) The origin. yes/no

   (d) A sphere of radius 1 in $\mathbb{R}^3$ centered at the origin. yes/no

   (e) A ball of radius 1 in $\mathbb{R}^3$ centered at the origin (this is the sphere and its interior) yes/no

   (f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$ yes/no

   (g) The null space of a 4×3 matrix. yes/no

   (h) The solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where A is a fixed 3×3 matrix and $\mathbf{b}$ is a fixed nonzero vector. yes/no

   (i) The column space of a 3×5 matrix. yes/no

   (j) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = xy$. yes/no

2. Determine if the statement is true or false. If it is true, give a proof. If it is false, find a counterexample.

   (a) If V is a nonzero vector space, then V contains infinitely many vectors.

   (b) If V has basis S and W is a subspace of V, then there exists a set T contained in S which is a basis for W.

   (c) If W is a subspace of V and T is a basis for W, then there exists a basis S for V which contains T.

   (d) If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a set of linearly independent vectors in a vector space V and $\mathbf{w}$ is a nonzero vector in V then the set $\{\mathbf{v_1}+\mathbf{w}, \mathbf{v_2}+\mathbf{w}, ..., \mathbf{v_k}+\mathbf{w}\}$ is also linearly independent.

   (e) If two matrices have the same RREF, then they have the same row space.

   (f) If two matrices have the same RREF, then they have the same column space.

(g) If U and W are subspaces of a vector space V and $\dim U < \dim W$, then U is a subspace of W.

(h) Any subspace of $\mathbb{R}^3$ which contains the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ must also contain the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(i) If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ is A is a fixed $m \times n$ matrix then the set $T = \{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^m$.

(j) If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ is A is a fixed invertible $n \times n$ matrix then the set $T = \{A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$.

(k) If V is spanned by an infinite set of vectors S, then V is infinite-dimensional.

(l) If V contains an infinite set of vectors which are linearly independent then V is infinite-dimensional.

(m) If W is a subspace of a finite dimensional vector space V and $\dim W = \dim V$ then $W = V$.

(n) Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. The set of solutions to the linear system $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$.

(o) Let V be a vector space with operations $\oplus, \odot$ and let $\mathbf{u}, \mathbf{v}$ be vectors in V and r a real number. If $r \odot (\mathbf{u} \oplus \mathbf{v}) = \mathbf{0}$ then either $r = 0$ or $\mathbf{v}$ is the negative of $\mathbf{u}$.

(p) Let V be a set with operations $\oplus$, $\odot$. Suppose there exists $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{u}$ and $\mathbf{w} \oplus \mathbf{v} \neq \mathbf{w}$. Then V is not a vector space.

(q) Let W be a subspace of V. If S is a set of linearly independent vectors in V and every vector in W can be written as a linear combination of the vectors in S then S is a basis for W.

(r) In an n-dimensional vector space V, any set consisting of at least n non-zero vectors is a spanning set for V.

(s) Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be a set of linearly independent vectors in a vector space V and $c \neq 0$ be a real number. Then the set $T = \{c\mathbf{v_1}, c\mathbf{v_2}, ..., c\mathbf{v_k}\}$ is linearly independent.

(t) Let S be a set of k vectors in V and let W be the span of S. Then $\dim W < k$ if and only if S is not a linearly independent set.

(u) Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be a set of vectors in a vector space V. Then any subspace of V which contains S also contains the span of S.

(v) Let W be the set of all polynomials whose coefficients add up to 0. W is an infinite dimensional subspace of P and W has basis $\{1 - t, 1 - t^2, 1 - t^3, 1 - t^4, ...\}$.

(w) If W is a subspace of a finite dimensional vector space V and $\dim W = \dim V$ then $W = V$.

(x) If W is a subspace of V and W and V are both infinite dimensional then $W = V$.