Math 3333-003
Spring 2016
Exam 2

Name: SOLUTIONS

Problem	Points
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Problem 6 (3 pts)	
Problem 7 (3 pts)	
Problem 8 (3 pts)	
Total	

Instructions:

- Calculators are allowed. You can use your calculator to put matrices in rref, find inverses, and multiply matrices.
- You must show all of your non-calculator work to receive full credit, unless explicitly stated otherwise.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Find a basis for the null space of

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

What is the dimension of the null space of A?

$$\text{rref}([A \mid \vec{0}]) = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4 free

$x_2 = s,\ x_4 = t$

$$\begin{aligned} x_1 &= -s - t \\ x_2 &= s \\ x_3 &= 0 \\ x_4 &= t \end{aligned} \quad , \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim = 2$$

2. (3 points) Find a basis for $\mathbb{R}^4$ which contains the vectors

$$\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\\3\end{bmatrix}.$$

Explain why your answer is a basis.

basis $= \left\{ \begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$

because $\begin{bmatrix}1 & 2 & 0 & 0\\1 & 1 & 0 & 0\\2 & 2 & 1 & 0\\3 & 3 & 0 & 1\end{bmatrix}$ is nonsingular

3. (3 points) Consider the vector space P_3 of all polynomials of degree less than or equal to 3. Let B be the basis

$$B = \{1, t+2, t^2+3t-1, t^3+1\}.$$

Find the polynomial f with

$$[f]_B = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix}.$$

$$f = 0 \cdot 1 + 2 \cdot (t+2) - 1 \cdot (t^2+3t-1) + 3 \cdot (t^3+1)$$

$$= 2t + 4 - t^2 - 3t + 1 + 3t^3 + 3$$

$$= 8 - t - t^2 + 3t^3$$

4. (3 points) Three different subspaces of $\mathbb{R}^4$ are given in a), b), c). In i), ii), iii) are the same three subspaces listed in a different order and described in a different way. Match them up with a), b), c). You do not need to show your work.

a) $\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\1\\1\\1\end{bmatrix}\right\}$

b) $\operatorname{span}\left\{\begin{bmatrix}-1\\2\\5\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\\0\end{bmatrix}, \begin{bmatrix}-7\\-11\\28\\11\end{bmatrix}, \begin{bmatrix}0\\-1\\5\\2\end{bmatrix}\right\}$

c) $\operatorname{span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}, \begin{bmatrix}9\\10\\11\\12\end{bmatrix}\right\}$

i) $\operatorname{span}\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$, this equals subspace a

ii) $\operatorname{span}\left\{\begin{bmatrix}-2\\-2\\-2\\-2\end{bmatrix}, \begin{bmatrix}2\\4\\6\\8\end{bmatrix}, \begin{bmatrix}0\\2\\4\\6\end{bmatrix}\right\}$, this equals subspace c

iii) $\operatorname{span}\left\{\begin{bmatrix}66\\66\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\66\\23\end{bmatrix}, \begin{bmatrix}0\\66\\0\\-17\end{bmatrix}\right\}$, this equals subspace b

5. (3 points) In each of the following items, a vector space V and a subset W is given. Circle Yes if the subset is a *subspace*; circle No if not. You do not need to show your work.

   (a) (Yes / No) $V = C(\mathbb{R}, \mathbb{R})$ (the vector space of all continuous functions) and $W = \{f \in V \mid f(1) + f(-2) = 0\}$.

   (b) (Yes / No) $V = C(\mathbb{R}, \mathbb{R})$ and $W = \{f \in V \mid f(2) = 5 + f(3)\}$.

   (c) (Yes / No) $V = \mathbb{R}^n$, A is an $n \times n$ matrix, and $W = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

   (d) (Yes / No) $V = \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^n$ is a nonzero vector, A is an $n \times n$ matrix, and $W = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b}\}$.

   (e) (Yes / No) $V = M_{n,n}$ (the vector space of $n \times n$ matrices), $\vec{b} \in \mathbb{R}^n$ is a nonzero vector, and $W = \{A \mid A\vec{b} = \vec{0}\}$.

   (f) (Yes / No) V a vector space, $\vec{v_1}, \vec{v_2}, \vec{v_3}$ vectors in V, and $W = \{x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} \mid x_1, x_2, x_3 \in \mathbb{R} \text{ with } x_1x_2x_3 = 0\}$.

6. (3 points) V is a vector space and $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis. Let $C = \{\vec{w_1}, \ldots, \vec{w_n}\}$ be a subset of V such that the set

$$S = \{[\vec{w_1}]_B, \ldots, [\vec{w_n}]_B\}$$

is a basis of $\mathbb{R}^n$. Prove that C is a basis of V. (Hint: You can use the fact $x[\vec{v}]_B + y[\vec{w}]_B = [x\vec{v} + y\vec{w}]_B$ for any scalars x, y and vectors $\vec{v}, \vec{w}$.)

Suppose $x_1 \vec{w_1} + \cdots + x_n \vec{w_n} = \vec{0}$

Then $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = [\vec{0}]_B = [x_1 \vec{w_1} + \cdots + x_n \vec{w_n}]_B$

$= x_1 [\vec{w_1}] + \cdots + x_n [\vec{w_n}]_B$

Since $[\vec{w_1}]_B, \ldots, [\vec{w_n}]_B$ is a basis of $\mathbb{R}^n$, this means that $x_1 = 0, \ldots, x_n = 0$.

Thus $\vec{w_1}, \ldots, \vec{w_n}$ are lin. ind. Since $\dim V = n$, they are also a basis.

7. (3 points) Let A be a 9×30 matrix. If $\dim\{A\vec{x} \mid \vec{x} \in \mathbb{R}^{30}\} = 7$, then:

(a) $\dim(\text{col space } A) =$ 7
Explain why.

$$\text{col sp } A = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^{30}\}$$

(b) $\dim(\text{row space } A) =$ 7
Explain why.

$$\dim(\text{col sp } A) = \dim(\text{row sp } A)$$

(c) $\dim(\text{null space } A) =$ 23
Explain why.

$$\dim(\text{null sp } A) + \dim(\text{col sp } A) = \#\text{ of cols of } A$$

8. (3 points)If A and B are square matrices of the same size, is it always true that

$$\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)?$$

If so, prove it. If not, give a counterexample.

No: $A = [1]$, $B = [-1]$

rank $A = 1$, rank $B = 1$

but rank $(A+B)$ = rank$([0]) = 0$.