Math 3333-001
Spring 2016
Exam 2

Name: SOLUTIONS

Problem	Points
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Problem 6 (3 pts)	
Problem 7 (3 pts)	
Problem 8 (3 pts)	
Total	

Instructions:

- Calculators are allowed. You can use your calculator to put matrices in rref, find inverses, and multiply matrices.
- You must show all of your non-calculator work to receive full credit, unless explicitly stated otherwise.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Find a basis for the null space of

$$A = \begin{bmatrix} 2 & 2 & 1 & 6 \\ 0 & 4 & 0 & 1 \end{bmatrix}.$$

What is the dimension of the null space of A?

$$\text{rref}([A \mid \vec{0}]) = \left[\begin{array}{cccc|c} 1 & 0 & .5 & 2.75 & 0 \\ 0 & 1 & 0 & .25 & 0 \end{array}\right]$$

x_3, x_4 free

$x_3 = s$, $x_4 = t$

$$\begin{aligned} x_1 &= -.5s - 2.75t \\ x_2 &= -.25t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -.5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2.75 \\ -.25 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} -.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.75 \\ -.25 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim = 2$

2. (3 points) Find a basis for $\mathbb{R}^4$ which contains the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \end{bmatrix}.$$

Explain why your answer is a basis.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

because $\operatorname{rref} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 1 \\ 3 & 3 & 0 & 0 \end{bmatrix} = I_4$

3. (3 points) Consider the vector space P_2 of all polynomials of degree less than or equal to 2. Let B be the basis

t^2+4t+4

$$B = \{1, t+2, (t+2)^2\}.$$

Find the coordinate vector of $1+t+t^2$ with respect to the basis B; that is, find

$$[1+t+t^2]_B.$$

$$t^2+t+1 = 1\cdot(t+2)^2 - 3\cdot(t+2) + 3\cdot 1$$

So $$[1+t+t^2]_B = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

4. (3 points) Three different subspaces of $\mathbb{R}^4$ are given in a), b), c). In i), ii), iii) are the same three subspaces listed in a different order and described in a different way. Match them up with a), b), c). You do not need to show your work.

a) $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

b) $\text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -11 \\ 28 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 5 \\ 2 \end{bmatrix} \right\}$

c) $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix} \right\}$

i) $\text{span}\left\{ \begin{bmatrix} -2 \\ -2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\}$, this equals subspace c

ii) $\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, this equals subspace a

iii) $\text{span}\left\{ \begin{bmatrix} 66 \\ 66 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 66 \\ 23 \end{bmatrix}, \begin{bmatrix} 0 \\ 66 \\ 0 \\ -17 \end{bmatrix} \right\}$, this equals subspace b

5. (3 points) In each of the following items, a vector space V and a subset W is given. Circle Yes if the subset is a *subspace*; circle No if not. You do not need to show your work.

   (a) (Yes / No) $V = P$ (the vector space of all polynomials) and $W = \{f \in P \mid f(1) = 0,\ f(2) = 0\}$.

   (b) (Yes / No) $V = P$ and $W = \{f \in P \mid f(-2) = -2\}$.

   (c) (Yes / No) $V = \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^n$ is a nonzero vector, A is an $n \times n$ matrix, and $W = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b}\}$.

   (d) (Yes / No) $V = \mathbb{R}^n$, A is an $n \times n$ matrix, and $W = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

   (e) (Yes / No) $V = M_{n,n}$ (the vector space of $n \times n$ matrices) and $W = \{A \mid A + 3A^T = I_n\}$.

   (f) (Yes / No) V a vector space, $\vec{v_1}, \vec{v_2}, \vec{v_3}$ vectors in V, and $W = \{x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} \mid x_1, x_2, x_3 \in \mathbb{R} \text{ with } x_1^2 + x_2^2 + x_3^2 \leq 1\}$.

6. (3 points) V is a vector space and $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$, $C = \{\vec{w_1}, \ldots, \vec{w_n}\}$ are two bases of V. Prove that the set

$$S = \{[\vec{w_1}]_B, \ldots, [\vec{w_n}]_B\}$$

is a basis of $\mathbb{R}^n$. (Hint: You can use the fact $x[\vec{v}]_B + y[\vec{w}]_B = [x\vec{v} + y\vec{w}]_B$ for any scalars x, y and vectors $\vec{v}, \vec{w}$.)

Suppose $x_1 [\vec{w_1}]_B + \cdots + x_n [\vec{w_n}]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Then $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = [x_1 \vec{w_1} + \cdots + x_n \vec{w_n}]_B$, so

$x_1 \vec{w_1} + \cdots + x_n \vec{w_n} = \vec{0}$. Since $\vec{w_1}, \ldots, \vec{w_n}$ is a basis, this means that $x_1 = 0, \ldots, x_n = 0$. So $[\vec{w_1}]_B, \ldots, [\vec{w_n}]_B$ are lin. ind. Since $\dim \mathbb{R}^n = n$, they form a basis.

7. (3 points) Let A be a 10×15 matrix. If $\dim\{A\vec{x} \mid \vec{x} \in \mathbb{R}^{15}\} = 7$, then:

(a) $\dim(\text{col space } A) =$ 7
Explain why.

col sp $A = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^{15}\}$

(b) $\dim(\text{row space } A) =$ 7
Explain why.

dim (col sp A) = dim (row sp A)

(c) $\dim(\text{null space } A) =$ 8
Explain why.

dim (null sp A) + dim (col sp A) = # of cols. of A

8. (3 points)If A and B are square matrices of the same size, is it always true that

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)?$$

If so, prove it. If not, give a counterexample.

No: $A = [1]$, $B = [-1]$

rank $A = 1$, rank $B = 1$

but rank $(A + B) =$ rank $([0]) = 0$