Differential Equations, Spring 2017
Worksheet, April 26, 2017

Goal: See how the geometry of vector fields and solutions of ODEs is controlled by eigenvalues and eigenvectors.

1. Consider the system
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
Use Matlab to find the eigenvalues and eigenvectors of the matrix. Remember that the command is something like (assuming the matrix is stored into the variable `A`)

```
>[P D]=eig(A)
```

The columnns of P are the eigenvectors and the diagonal entries of D are the eigenvalues.

2. Let $\vec{v_1}, \vec{v_2}$ be the columns of P and λ_1, λ_2 be the eigenvalues. Then the general solution is
$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$$
Convince yourself (say by multiplying out or by thinking of P as $P = [\vec{v_1}\ \vec{v_2}]$) that this is the same thing as
$$\vec{x}(t) = P \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}.$$

3. In particular,
$$\vec{x}(0) = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$
So
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\vec{x}(0).$$
Suppose the IC is $\vec{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Use the previous formula to find c_1, c_2. For example, entering

```
c=inv(P)*[-2;3]
```

will do this.

4. Write down the solution to the IVP. Hint: The following command will make Matlab display $c_1\vec{v_1}$:

```
c(1)*P(:,1)
```

A similar command will work for $c_2\vec{v_2}$.

5. Now code the right hand side of the ODE (that is, the vector field $\vec{G}(t, \vec{x}) = A\vec{x}$) into a Matlab function. We want to be able to pass this function into the Improved Euler Method, so it should have the following form:

```
function val=G(t,x)

end
```

6. Use the Improved Euler Method to solve the IVP, and then graph the solution in the x_1x_2-plane. For the time range, use something like $0 \le t \le 1$. (If you use a large time range the solution will shoot off to infinity and your graph will look like a straight line because it will be very zoomed out.) Store the commands to do this in a script called `graphs.m`. Convince yourself that the graph is consistent with your answer from number 4.

7. Solve (using the Improved Euler Method) and graph (on the same axes) the solutions for several other IVPs. For some of them use eigenvectors as the IC. Put all the commands into the script `graphs.m`. This will make your life easier – if you make a mistake, you can just edit and rerun the script instead of typing in all the commands again. Also, experiment with different time ranges and ICs so that you get a picture that looks nice. You will need to have the command `hold on` near the beginning to tell Matlab to draw everything on the same figure. You may want to do the next problem first before spending too much time on this one.

8. Graph the vector field $\vec{G}(t, \vec{x})$. Here is some code from class on April 3rd that we used to graph a vector field. Edit it as needed to graph the vector field for `G.m`.

```
xmin=-1;
xmax=1;
ymin=-2;
ymax=2;
xsteps=10;
ysteps=10;
u=[];
v=[];
X=[];
Y=[];
%% generate data for vector field
x=linspace(xmin,xmax,xsteps);
y=linspace(ymin,ymax,ysteps);
for i=x
for j=y
X=[X i];
Y=[Y j];
```

```
vector=G(0,[i;j]);
u=[u vector(1)];
v=[v vector(2)];
end
end
%% draw vector field
fig1=figure;
hold on;
quiver(X,Y,u,v);
```

You can put the code into your script `graphs.m`. If you want it to draw on the same figure as the previous plots, remove the `fig1=figure` command. You might want to change the `xmin,xmax` etc. in order to get a nice looking picture.

9. You can also try to animate one of your solutions. See the code in `dampedDemo.m` from the April 3rd class for an example of how to do this.

10. The matrix for the above problems had one positive eigenvalue and one negative eigenvalue. It is instructive to repeat the above problems (at least graph the vector fields) for the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{(two positive eigenvalues)}$$

$$\begin{bmatrix} 4 & -2 \\ 15 & -7 \end{bmatrix} \qquad \text{(two negative eigenvalues)}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{(purely imaginary eigenvalues, undamped spring for example)}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{(complex eigenvalues with positive real part)}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \text{(complex eigenvalues with negative real part, underdamped spring for example)}$$