Differential Equations, Spring 2017
Written Assignment #5, Due Friday, April 28

The goal of this assignment is to use matrix exponentials to solve some ODEs. If A is a square $n \times n$ matrix, then the exponential of A is the $n \times n$ matrix defined by

$$e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

In other words: we just use the power series definition of the exponential function! (Warning: In general, $e^A e^B \neq e^{A+B}$. If $AB = BA$, it is however true that $e^A e^B = e^{A+B}$.) If t is a scalar variable, then the exponential of tA is the function

$$\begin{aligned} e^{tA} &= I_n + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{+}\cdots \\ &= I_n + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{t^kA^k}{k!}. \end{aligned}$$

The derivative of e^{tA} is Ae^{tA}, because

$$\begin{aligned} \frac{d}{dt}e^{tA} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^kA^k}{k!} = \sum_{k=0}^{\infty}\frac{d}{dt}\left(\frac{t^kA^k}{k!}\right) = \sum_{k=1}^{\infty}k\frac{t^{k-1}A^k}{k!} = \sum_{k=1}^{\infty}\frac{t^{k-1}A^k}{(k-1)!} \\ &= A\sum_{k=1}^{\infty}\frac{t^{k-1}A^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty}\frac{t^kA^k}{k!} = Ae^{tA}. \end{aligned}$$

We can use this property to easily solve a homogeneous linear ODE: The solution to the IVP

$$\begin{aligned} \vec{x}' &= A\vec{x}, \\ \vec{x}(0) &= \vec{x_0} \end{aligned}$$

is

$$\vec{x}(t) = e^{tA}\vec{x_0}$$

because

$$\frac{d}{dt}\vec{x}(t) = \left(\frac{d}{dt}e^{tA}\right)\vec{x_0} = Ae^{tA}\vec{x_0} = A\vec{x}(t), \quad \vec{x}(0) = e^{0A}\vec{x_0} = I_n\vec{x_0} = \vec{x_0}.$$

We will call this method of solution the *matrix exponential method.* Now onto the problems.

1. We begin with the IVP

$$\begin{aligned} x_1' &= -x_2, \\ x_2' &= x_1, \\ x_1(0) &= c_1, \\ x_2(0) &= c_2. \end{aligned}$$

In matrix notation, this is the equation $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(a) Show that

$$e^{tA} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Hint: First compute the powers $(tA)^k$. Try to recognize a pattern. Then write down e^{tA}, and use the power series of $\cos t$ and $\sin t$ to simplify it.

(b) Use part (a) and the matrix exponential method (explained in the introduction) to write down the general solution of the IVP. (Note: Since the IC involves the arbitrary constants c_1, c_2, this will be the general solution.)

(c) Use the methods of Chapter 4 (elimination/substitution) to solve the IVP. Check that your answer agrees with part (b).

Remark: Part (a) is basically a matrix analog of Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ because

$$e^{tA} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = I_2 \cos t + A \sin t.$$

The matrix A plays the role of multiplication by i. In fact, if we identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ in the standard way, then the transformation $\vec{v} \mapsto A\vec{v}$ is identified with multiplication by i.

2. Next consider the system

$$\begin{aligned} x_1' &= x_1 + x_2, \\ x_2' &= x_2, \\ x_1(0) &= c_1, \\ x_2(0) &= c_2. \end{aligned}$$

In matrix notation, this is the equation $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(a) Show that

$$e^{tA} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

(b) Use part (a) and the matrix exponential method to write down the general solution of the IVP. (Note: Since the IC involves the arbitrary constants c_1, c_2, this will be the general solution.)

(c) Use the methods of Chapter 4 (elimination/substitution) to solve the IVP. Check that your answer agrees with part (b).

Remark: This problem explains why an extra t factor appears in one of the solutions for a (1 dim) second order ODE with a double characteristic root, and also why correction factors for particular solutions are of the form t^d.

3. Finally, let's relate the matrix exponential method to the eigenvalue method.

   (a) Suppose λ is an eigenvalue of A with eigenvector $\vec{v}$. Show that $e^{t\lambda}$ is an eigenvalue of e^{tA} with eigenvector $\vec{v}$. Hint: Compute $e^{tA}\vec{v}$.

   (b) Continuing part (a), let $\vec{x}(t) = e^{tA}\vec{v}$. By the matrix exponential method, we know this is a solution of the IVP $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{v}$. How does this solution relate to the solution produced by the eigenvalue method? Hint: There is almost no work that needs to be done for this problem (beyond part (a)).

   (c) Now consider the following general type of situation: Suppose the $n \times n$ matrix A has n linearly independent eignvectors $\vec{v_1}, \ldots, \vec{v_n}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. An arbitrary vector $\vec{v} \in \mathbb{R}^n$ can be written as $\vec{v} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$ for some scalars $c_1, \ldots, c_n$. Show that the solution $\vec{x}(t) = e^{tA}\vec{v}$ to the IVP $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{v}$ produced by the matrix exponential method agrees with the solution produced by the eigenvalue method.