Homework 5 Solution

1. a) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = A \cdot A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A^6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

...

So $e^{tA} = I_2 + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \cdots$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + 0 - t^2/2! + 0 + t^4/4! + \cdots & 0 - t + 0 + t^3/3! + \cdots \\ 0 + t + 0 - t^3/3! + 0 + \cdots & 1 + 0 - t^2/2! + 0 + t^4/4! + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

b) $\vec{X}(t) = e^{tA}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\vec{X} = \begin{bmatrix} c_1 \cos t - c_2 \sin t \\ c_1 \sin t + c_2 \cos t \end{bmatrix}$$

c) $\left[\begin{array}{l} x_1' = -x_2 \\ x_2' = x_1 \\ x_1(0) = c_1 \\ x_2(0) = c_2 \end{array}\right. \Longrightarrow \quad x_2'' = x_1' \nearrow$

$x_2'' = -x_2$

$x_2'' + x_2 = 0$

$r^2 + 1 = 0, \quad r = \pm i$

$\Rightarrow \quad x_2 = C_2 \cos t + C_1 \sin t$

then $\quad x_1 = x_2' = C_1 \cos t - c_2 \sin t$

So $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t - c_2 \sin t \\ c_1 \sin t + c_2 \cos t \end{bmatrix}$

so agrees with part b)

2. a) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad$ So $A^K = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\vdots$

$$e^{tA} = I_2 + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & t \\ 0 & t \end{bmatrix} + \begin{bmatrix} t^2/2! & 2 \cdot t^2/2! \\ 0 & t^2/2! \end{bmatrix} + \begin{bmatrix} t^3/3! & 3 \cdot t^3/3! \\ 0 & t^3/3! \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + t + t^2/2! + t^3/3! + \cdots & t + 2\cdot t^2/2! + 3\cdot t^3/3! + \cdots \\ 0 & 1 + t + t^2/2! + t^3/3! + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^t & t(1 + 2\cdot t/2 + 3\cdot t^2/3! + \cdots) \\ 0 & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t & t(1 + t + t^2/2! + \cdots) \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

b) $$\vec{x} = e^{tA}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 te^t \\ c_2 e^t \end{bmatrix}$$

c) $$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_2 \\ x_1(0) = c_1 \\ x_2(0) = c_2 \end{cases} \Rightarrow x_2 = c_2 e^t \Rightarrow x_1' - x_1 = c_2 e^t$$

char. eqtn. $r - 1 = 0$

$x_{1,c} = De^t$

$x_{1,p} = Ate^t$

$x_{1,p}' = Ate^t + Ae^t$

$\Rightarrow Ate^t + Ae^t - Ate^t = c_2 e^t$

$\Rightarrow A = c_2$

$\Rightarrow x_{1,p} = c_2 te^t$

$\Rightarrow x_1 = x_{1,c} + x_{1,p} = D_1 e^t + c_2 te^t$

Need $x_1(0) = c_1$

So $D_1 e^0 + c_2 0 e^0 = c_1 \Rightarrow D_1 = c_1$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= c_1 e^t + c_2 t e^t \\ x_2 &= c_2 e^t \end{aligned}}$$

Agrees with part b).

3. a) $A\vec{v} = \lambda \vec{v}$

$$A^2 \vec{v} = A(A\vec{v}) = A(\lambda \vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2 \vec{v}$$

$$A^3 \vec{v} = \lambda^3 \vec{v}$$

$$\vdots$$

So $$e^{tA}\vec{v} = \left(I_n + tA + \frac{t^2}{2!}A^2 + \cdots\right)\vec{v}$$

$$= I_n\vec{v} + tA\vec{v} + \frac{t^2}{2!}A^2\vec{v} + \cdots$$

$$= \vec{v} + t\lambda\vec{v} + \frac{t^2\lambda^2}{2!}\vec{v} + \cdots$$

$$= \left(1 + t\lambda + \frac{t^2\lambda^2}{2!} + \cdots\right)\vec{v} = e^{\lambda t}\vec{v}$$

b) Eigenvalue method: $\vec{x} = e^{\lambda t}\vec{v}$ is a solution

Matrix exponential method: $\vec{x} = e^{At}\vec{v}$ is a solution

By part a), $\vec{x} = e^{At}\vec{v} = e^{\lambda t}\vec{v}$.

So the two methods produce the same solution.

c) By eigenvalue method, general solution is

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v_1} + \cdots + c_n e^{\lambda_n t} \vec{v_n}$$

(notice $\vec{x}(0) = c_1 \vec{v_1} + \cdots + c_n \vec{v_n} = \vec{v}$)

By matrix exponential method, general solution is

$$\vec{x} = e^{At} \vec{v}$$

since $\vec{v} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$, this is

$$\begin{aligned} e^{At}\vec{v} &= e^{At}\left(c_1\vec{v_1} + \cdots + c_n\vec{v_n}\right) \\ &= c_1 e^{At}\vec{v_1} + \cdots + c_n e^{At}\vec{v_n} \\ &= c_1 e^{\lambda_1 t}\vec{v_1} + \cdots + c_n e^{\lambda_n t}\vec{v_n} \end{aligned}$$

So the two methods produce the same solution.