Math 3113
Spring 2017
Exam 2, March 24

Name: SOLUTIONS

Problem	**Points**
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Total	

Instructions:

- Calculators are allowed. No cell phones allowed.
- You must show all of your work to receive credit.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Solve the initial value problem

$$\begin{aligned} y'' + 2y' + y &= 0, \\ y(0) &= 0, \\ y'(0) &= 2. \end{aligned}$$

char eqtn: $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \Rightarrow r = -1$ is double root

general soltn is $y = c_1 e^{-x} + c_2 x e^{-x}$

$$y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$$

$$= (-c_1 + c_2) e^{-x} - c_2 x e^{-x}$$

IC:

$$0 = y(0) = c_1$$

$$2 = y'(0) = -c_1 + c_2 \Rightarrow c_1 = 0, \quad c_2 = 2$$

$$\boxed{y = 2 x e^{-x}}$$

2. (3 points) Find the general solutions.

(a) $y^{(4)} + y^{(3)} = 0$

triple root

$r^4 + r^3 = 0$

$r^3(r+1) = 0 \Rightarrow r = 0, -1$

$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x}$

(b) $y^{(4)} + 2y^{(2)} + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$r = \pm i$, each double root

$y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x$

(c) $y'' + y' + y = 0$

$r^2 + r + 1 = 0$

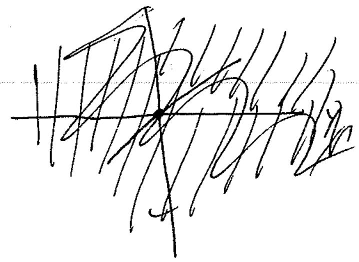
$r = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

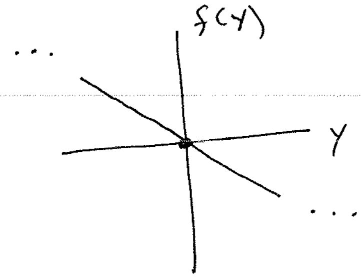
$y = e^{-\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

3. (3 points) Consider the autonomous ODE $y' = f(y)$. Suppose $f(0) = 0$. Draw a few pictures and write a few sentences to explain why the following statement is true: If $f'(0) < 0$, then $y = 0$ is a stable equilibrium solution.

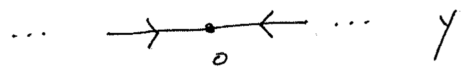
$f(0) = 0$, $f'(0) < 0$ $\Rightarrow$ near $y=0$, the graph of $f(y)$

looks like :





So the phase diagram near $y=0$ looks like:



Thus $y=0$ is a stable equilibrium solution.

4. (3 points) Find an implicit general solution to

$$\frac{dy}{dx} = -\frac{x + 2x^2y}{x^3 + xe^y}.$$

(Hint: It may be helpful to simplify the right hand side a little bit first.)

$$\frac{dy}{dx} = -\frac{x(1+2xy)}{x(x^2+e^y)} = -\frac{1+2xy}{x^2+e^y}$$

$$\Rightarrow \quad (x^2+e^y)\,dy = -(1+2xy)\,dx$$

$$(1+2xy)\,dx + (x^2+e^y)\,dy = 0$$

$$\downarrow \partial/\partial y \qquad\qquad \downarrow \partial/\partial x$$

$$2x \qquad\qquad 2x$$

So is exact. Find H with

$$\begin{cases} \dfrac{\partial H}{\partial x} = 1+2xy \\ \dfrac{\partial H}{\partial y} = x^2+e^y \end{cases}$$

$$H = \int \frac{\partial H}{\partial x}\,dx = \int (1+2xy)\,dx = x + x^2y + C(y)$$

$$\frac{\partial H}{\partial y} = x^2 + C'(y) = x^2 + e^y \quad \Rightarrow \quad C'(y) = e^y \Rightarrow C(y) = e^y$$

So solution is $\boxed{x + x^2y + e^y = C}$

5. (3 points) A spring with spring constant $k = 5$ N/m has a mass of m kg attached. The spring is damped by friction. Assume the damping constant c has the form $c = 5m$ (m is the mass). So the ODE is

$$mx'' + 5mx' + 5x = 0.$$

The spring is then put into motion. It is observed that the resulting motion is underdamped and has an angular frequency of 10 radians per second. What is the mass?

char eqn. $mr^2 + 5mr + 5 = 0$

this is < 0 b/c underdamped

$$r = \frac{-5m \pm \sqrt{25m^2 - 20m}}{2m}$$

$$= -\frac{5}{2} \pm i \underbrace{\frac{\sqrt{20m - 25m^2}}{2m}}$$

angular freq.

$$\Rightarrow \quad 10 = \frac{\sqrt{20m - 25m^2}}{2m}$$

$$100 = \frac{20m - 25m^2}{4m^2}$$

$$400m^2 = 20m - 25m^2$$

$$425m^2 = 20m \Rightarrow \boxed{m = \frac{20}{425} \text{ kg}}$$