Section 2.2

21

$$\frac{dx}{dt} = Kx - x^3$$

$$f(x, K) = Kx - x^3$$

a) crit. values satisfy $f(x,K) = 0$

so $\quad Kx - x^3 = 0$

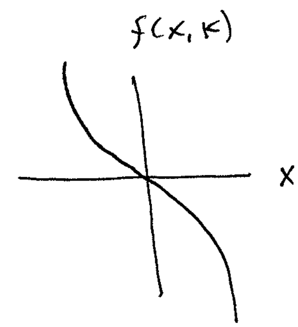
$$x(K - x^2) = 0$$

for $K \leq 0$ only crit. value is $x = 0$

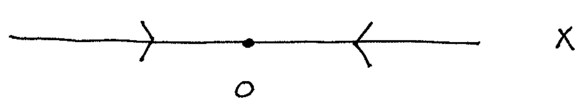
for $K \geq 0$ crit. values are $x = 0, \pm\sqrt{K}$

For $K \leq 0$, $f(x,K) \geqslant 0$ when $x < 0$

and $f(x,K) \leqslant 0$ when $x > 0$

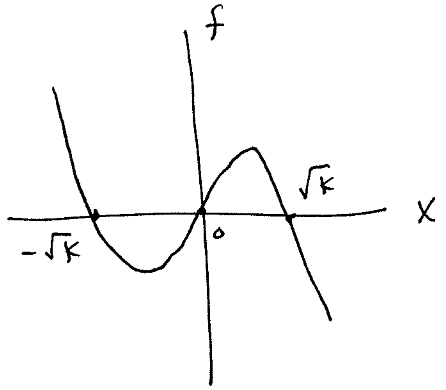


so the phase diagram is

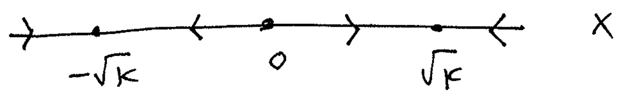


So $x = 0$ is a stable critical point

b) For $k>0$, the graph of $f(x,k)$ vs. x is



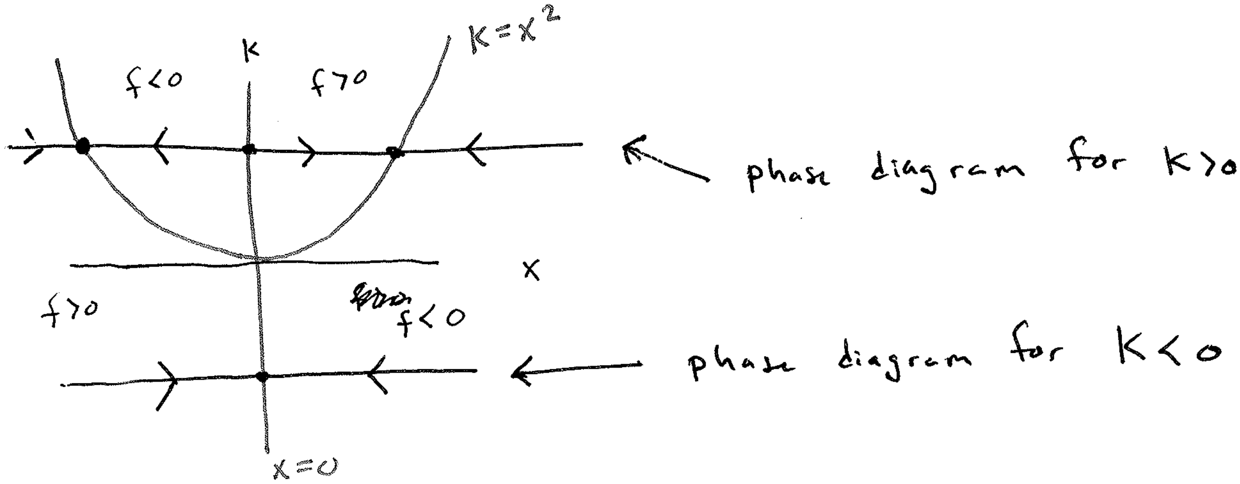
So the phase diagram is



So $\pm\sqrt{k}$ are stable

and 0 is unstable

Remark: We can see all phase diagrams at once by imagining plotting them on the bifurcation diagram (graph of $f(x,k)=0$ in xk-plane).



($f(x,k)=0 \iff x=0$ or $k=x^2$)

#23

a) $$\frac{dx}{dt} = kx(M-x) - hx$$

$$= kMx - kx^2 - hx$$

$$= (kM-h)x - kx^2$$

$$= k\left[\frac{kM-h}{k}x - x^2\right]$$

$$= kx\left[\frac{kM-h}{k} - x\right]$$

$$= kx(M_1 - x) \quad \text{with} \quad M_1 = \frac{kM-h}{k}$$

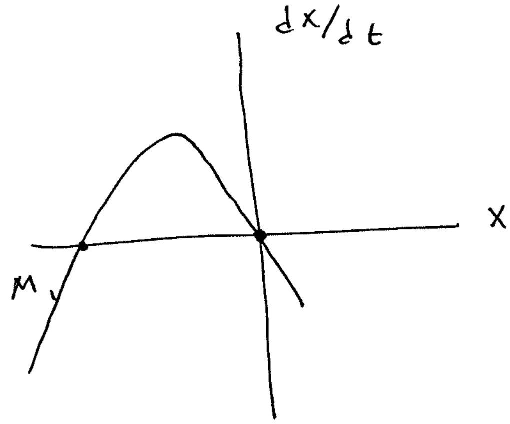
This is logistic with new carrying capacity

$$M_1 = \frac{kM-h}{k} \quad \left(\text{as long as } kM-h > 0\right)$$

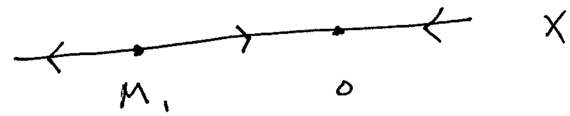
Thus the limiting population is $\frac{kM-h}{k}$

b) From a), $\frac{dx}{dt} = kx(M_1 - x)$.

If $h \geq kM$ then $M_1 \leq 0$ and the graph of $\dfrac{dx}{dt}$ vs. x is



Thus the phase diagram is



For fish, only $x_0 \geq 0$ makes sense, and we see that

$$\lim_{t\to\infty} x(t) = 0.$$

#24

$$\begin{cases} \dfrac{dx}{dt} = K(N-x)(x-H) \\ x(0) = x_0 \end{cases}$$

First note that $\dfrac{1}{(N-x)(x-H)} = \dfrac{(N-H)^{-1}}{N-x} + \dfrac{(N-H)^{-1}}{x-H}$.

So $$\int \frac{dx}{(N-x)(x-H)} = \int K\,dt$$

$$-(N-H)^{-1}\ln|N-x| + (N-H)^{-1}\ln|x-H| = Kt + C$$

$$(N-H)^{-1}\ln\left|\frac{x-H}{N-x}\right| = Kt + C$$

$$\ln\left|\frac{x-H}{N-x}\right| = (N-H)(Kt+C)$$

$$\frac{X - H}{N - X} = C e^{(N-H)kt} \quad \text{, use IC:} \quad \frac{X_0 - H}{N - X_0} = C$$

$$X\left(1 + C e^{(N-H)kt}\right) = N C e^{(N-H)kt} + H$$

$$X = \frac{N C e^{(N-H)kt} + H}{1 + C e^{(N-H)kt}}$$

$$X = \frac{N\left(\frac{X_0 - H}{N - X_0}\right) e^{(N-H)kt} + H}{1 + \left(\frac{X_0 - H}{N - X_0}\right) e^{(N-H)kt}} \cdot \frac{e^{-(N-H)kt}\,(N - X_0)}{e^{-(N-H)kt}\,(N - X_0)}$$

$$X = \frac{N(X_0 - H) + H(N - X_0)\, e^{-(N-H)kt}}{(N - X_0)\, e^{-(N-H)kt} + (X_0 - H)}$$

Section 2.3

27

Remark: To derive eqtn. $\frac{dr}{dt} = \frac{K}{\sqrt{r}}$:

a) $r_0 = R$, $v_0 = \sqrt{\frac{2GM}{R}}$

Conservation of energy eqtn. is

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

using IC, $E = \frac{1}{2}m \cdot \frac{2GM}{R} - \frac{GmM}{R} = 0.$

Thus $0 = \frac{1}{2}mv^2 - \frac{GmM}{r}$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{GmM}{r}$$

$$\Rightarrow \quad v^2 = \frac{2MG}{r}$$

$$v = \frac{\sqrt{2MG}}{\sqrt{r}}$$

$$\frac{dr}{dt} = \frac{K}{\sqrt{r}} \quad \text{where} \quad K = \sqrt{2MG}.$$

Now we have IVP $\left[\begin{array}{l} \frac{dr}{dt} = \frac{k}{\sqrt{r}} \\ r_0 = R \end{array}\right.$

Separate variables: $\sqrt{r}\, dr = K\, dt$

$$\Rightarrow \quad \int \sqrt{r}\, dr = \int K \, dt$$

$$\frac{2}{3} r^{3/2} = Kt + C$$

$$r^{3/2} = \frac{3K}{2} t + C$$

$$r = \left(\frac{3K}{2} t + C \right)^{2/3}$$

IC : $\quad R = r(0) = C^{2/3}, \quad C = R^{3/2}$

so $\quad r(t) = \left(\frac{3K}{2} t + R^{3/2} \right)^{2/3}$

So $\displaystyle \lim_{t \to \infty} r(t) = \lim_{t \to \infty} \left(\frac{3K}{2} t + R^{3/2} \right)^{2/3} = \infty .$

b) If $v_0 > \sqrt{\frac{2GM}{R}}$, then energy is

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{R} > \frac{1}{2} m \left(\frac{2GM}{R} \right) - \frac{GMm}{R} = 0$$

So $E > 0$, and

$$E = \frac{1}{2} m v^2 - \frac{GmM}{r}$$

$$\Rightarrow \quad \frac{1}{2} m v^2 = E + \frac{GmM}{r}$$

$$v^2 = \frac{2E}{m} + \frac{2GM}{r} = \alpha + \frac{K^2}{r} \quad \text{where } \alpha = \frac{2E}{m} > 0$$

$$v = \sqrt{\alpha + \frac{K^2}{r}},$$

So $\frac{dr}{dt} = v = \sqrt{\alpha + \frac{k^2}{r}} > \sqrt{\frac{k^2}{r}} = \frac{k}{\sqrt{r}}$.

Suppose r stays finite, say $r \leq N$.

Then $\frac{dr}{dt} > \frac{k}{\sqrt{r}} \geq \frac{k}{\sqrt{N}}$,

i.e. velocity $\geq \frac{k}{\sqrt{N}}$ at <u>all</u> times.

Thus $r(t) \geq r_0 + t\frac{k}{\sqrt{N}}$ and it follows that $\lim_{t\to\infty} r(t) \geq \lim_{t\to\infty} r_0 + t\frac{k}{\sqrt{N}} = \infty$.