Math 3113-007
Fall 2016 Exam 1

Name: SOLUTIONS

Problem	Points
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Total	

Instructions:

- Calculators are allowed. No cell phones allowed.
- You must show all of your work to receive credit.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Solve the initial value problem

$$\frac{dy}{dx} = y + x$$
$$y(1) = 2$$

$\frac{dy}{dx} - y = x$, integ. factor $= e^{\int -dx} = e^{-x}$

$\Rightarrow \quad \frac{d}{dx}\left(ye^{-x}\right) = xe^{-x}$

$ye^{-x} = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

$y = -x - 1 + Ce^{x}$

$y(1) = 2 \Rightarrow \quad 2 = -1 - 1 + Ce, \quad C = 4e^{-1}$

$\Rightarrow \quad \boxed{y = -x - 1 + 4e^{x-1}}$

2. (3 points) Find the general solution of the equation

$$y' = x(1 + y^2).$$

Your answer should express y as a function of x.

$$\frac{dy}{dx} = x(1+y^2)$$

$$\frac{dy}{1+y^2} = x\,dx$$

$$\int \frac{dy}{1+y^2} = \int x\,dx$$

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \tan\left(\frac{1}{2}x^2 + C\right)}$$

3. (3 points) Consider the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \\ y(0) &= 1 \end{aligned}$$

Use Euler's method with a step size of .25 to approximate $y(1)$. How far away is your approximation from the exact value of $y(1)$?

$f(x,y) = 3x^2$

$y_{i+1} = y_i + hf(x_i, y_i)$

$h = .25$

i	x_i	y_i	$f(x_i, y_i)$
0	0	1	0
1	.25	1	.1875
2	.5	1.046875	.75
3	.75	1.234375	1.6875
4	1	1.65625	

$\boxed{y(1) \approx 1.65625}$

the exact value of $y(1)$ is

$$y(1) = 1 + \int_0^1 3x^2\,dx = 1 + x^3\Big|_0^1 = 2$$

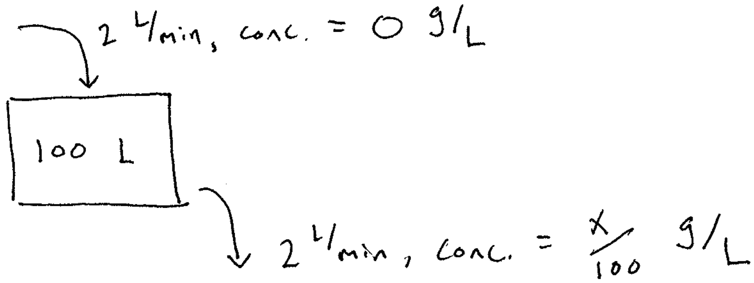
The difference is $\boxed{2 - 1.65625 = .34375}$

4. (3 points) Mixture problem! You do not need to explicitly solve this problem, just set it up. A tank initially contains 100 liters of a water and salt solution that has a concentration of 5 grams per liter. Pure water starts to flow into the tank at a rate of 2 liters per minute. Thoroughly mixed water flows out of the tank at the same rate. Let $x(t)$ denote the amount of salt inside the tank at time t. x is measured in grams and t is measured in minutes.

   (a) Write down the differential equation and initial condition that $x(t)$ satisfies. Do not solve the equation.

   (b) Draw either a slope field or a phase diagram for the differential equation (you can pick which one you want to draw). What is the long term behavior of $x(t)$, i.e. what is $\lim_{t\to\infty} x(t)$?

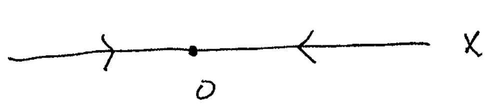
a)



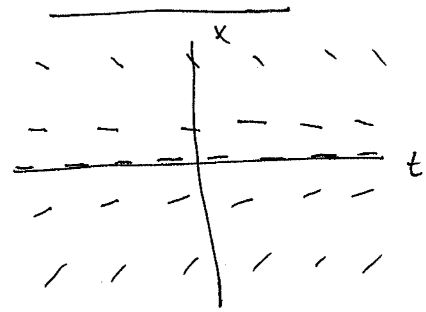
initial amount = (100 L)(5 g/L) = 500 g

$$\boxed{\begin{array}{l} \dfrac{dx}{dt} = -\dfrac{2x}{100} \\[2ex] x(0) = 500 \end{array}}$$

b) phase diagram



Slope field



$$\boxed{\lim_{t\to\infty} x(t) = 0}$$

5. (3 points) Recall that Newton's law of cooling and heating for an object in an environment says that

$$\frac{dT}{dt} = k(T - A)$$

where k is a constant, T is the temperature of the object, and A is the temperature of the environment (a constant).

(a) Is there any constraint on the sign of the constant k? In other words, does k have to be positive or negative, or can it be either? Explain your answer.

(b) Suppose that a cup of water with an initial temperature of 20° C is put outside where the temperature is −30° C. Assume that t is measured in minutes. What are the units on k?

(c) Continuing the previous question, suppose it is found that the temperature of the cup of water drops by 1° C during the first 10 seconds it is outside. Without solving the differential equation, give an approximate value for k. Explain your reasoning.

a) $\boxed{K < 0}$ because: if $T(t)$ is bigger than A then $\frac{dT}{dt}(t) < 0$ and $T - A > 0$, so K will be negative. Similar logic holds if $T(t) < A$.

b) $\frac{dT}{dt}$ has units °C/min, $(T-A)$ has units °C,

so K has units 1/min.

c) rate of change of temp during first 10 seconds is

$$-\frac{1°\,C}{10\text{ sec}} = \frac{-1°C}{\frac{1}{6}\text{ min}} = -6°\,C/\text{min}, \text{ so } T'(0) \approx -6°\,C/\text{min}$$

and $K \approx \frac{T'(0)}{T(0) - A}$

$$= -\frac{6°\,C/\text{min}}{20°C + 30°C} = -\frac{6}{50}\ 1/\text{min} = \boxed{-.12\ 1/\text{min}}$$