Math 3113-004
Fall 2016 Exam 1

Name: SOLUTIONS

Problem	Points
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Total	

Instructions:

- Calculators are allowed. No cell phones allowed.
- You must show all of your work to receive credit.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Solve the initial value problem

$$\frac{dy}{dx} = y + 2x$$
$$y(0) = 3$$

$\frac{dy}{dx} - y = 2x,$ integ. factor $= e^{\int -dx} = e^{-x}$

$\frac{d}{dx}\left(ye^{-x}\right) = 2xe^{-x}$

$ye^{-x} = -2xe^{-x} - 2e^{-x} + C$

$y = -2x - 2 + Ce^{x}$

$y(0) = 3 \Rightarrow 3 = -2 + C, \quad C = 5$

$\boxed{y = -2x - 2 + 5e^{x}}$

2. (3 points) Find the general solution of the equation

$$y' = y^2.$$

Your answer should express y as a function of x.

$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int dx$$

$$-y^{-1} = x + C$$

$$\boxed{y = \frac{1}{-x + C}}$$

3. (3 points) Consider the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= -3x^2 \\ y(0) &= 4 \end{aligned}$$

Use Euler's method with a step size of .5 to approximate $y(2)$. How far away is your approximation from the exact value of $y(2)$?

$h = .5, \quad f(x,y) = -3x^2$

$y_{i+1} = y_i + h f(x_i, y_i)$

i	x_i	y_i	$f(x_i, y_i)$
0	0	4	0
1	.5	4	-.75
2	1	3.625	-3
3	1.5	2.125	-6.75
4	2	-1.25	

$\boxed{y(2) \approx -1.25}$

exact answer: $y(2) = 4 + \int_0^2 -3x^2\,dx = 4 - x^3 \Big|_0^2 = -4$

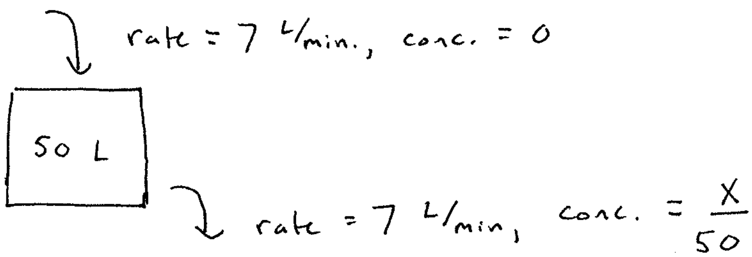
$\boxed{\text{difference is } |-4 + 1.25| = 2.75}$

4. (3 points) Mixture problem! You do not need to explicitly solve this problem, just set it up. A tank initially contains 50 liters of a water and salt solution that has a concentration of 2 grams per liter. Pure water starts to flow into the tank at a rate of 7 liters per minute. Thoroughly mixed water flows out of the tank at the same rate. Let $x(t)$ denote the amount of salt inside the tank at time t. x is measured in grams and t is measured in minutes.

   (a) Write down the differential equation and initial condition that $x(t)$ satisfies. Do not solve the equation.

   (b) Draw either a slope field or a phase diagram for the differential equation (you can pick which one you want to draw). What is the long term behavior of $x(t)$, i.e. what is $\lim_{t\to\infty} x(t)$?

a)

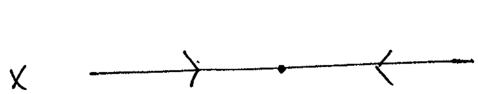


initial amount $= (50\text{ L})(2\text{ g/L}) = 100\text{ g}$

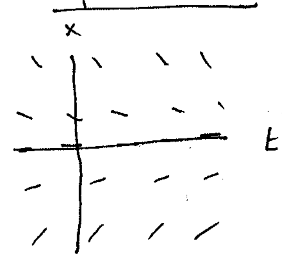
$$\boxed{\frac{dx}{dt} = 7\cdot 0 - 7\cdot\frac{x}{50} = -\frac{7}{50}x \qquad x(0) = 100}$$

b)

phase diagram



Slope field



$$\boxed{\lim_{t\to\infty} x(t) = 0}$$

as seen by the pictures

5. (3 points) Recall that Newton's law of cooling and heating for an object in an environment says that

$$\frac{dT}{dt} = k(A - T)$$

where k is a constant, T is the temperature of the object, and A is the temperature of the environment (a constant).

(a) Is there any constraint on the sign of the constant k? In other words, does k have to be positive or negative, or can it be either? Explain your answer.

(b) Suppose that a cup of water with an initial temperature of 20° C is put outside where the temperature is −30° C. Assume that t is measured in minutes. What are the units on k?

(c) Continuing the previous question, suppose it is found that the temperature of the cup of water drops by 1° C during the first 5 seconds it is outside. Without solving the differential equation, give an approximate value for k. Explain your reasoning.

a) $\boxed{K > 0}$: If $T > A$ then the object will cool, so $\frac{dT}{dt} < 0$.

By the eqtn., $K = \frac{dT/dt}{A - T} = \frac{\text{negative}}{\text{negative}} > 0$.

Similar logic holds if $T < A$.

b) $K = \frac{dT/dt}{A - T}$ has units $\frac{°C/\text{min}}{°C} = \boxed{1/\text{min}.}$

c) temp. drops by 1°C during first 5 seconds

$\Rightarrow T'(0) \approx -\frac{1°C}{5 \text{ sec}} = -\frac{1°C}{\frac{5}{60}\text{ min}} = -.12 \; °C/\text{min}.$

Then $K = \frac{T'(0)}{A - T(0)} \approx \frac{-12 \; °C/\text{min}}{-50° \, C} = \frac{12}{50} \; 1/\text{min} = \frac{6}{25} \; 1/\text{min}$

$= .24 \; 1/\text{min}$