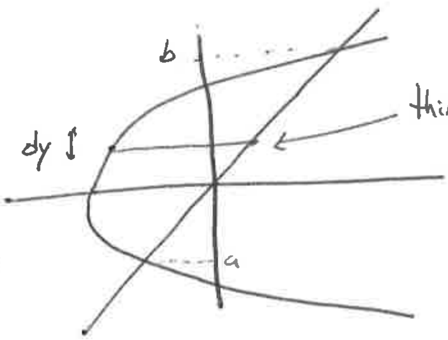
Differential and Integral Calculus 2, Math 2924-050, Fall 2014
Practice Exam 1

1. Find the area of the bounded region contained between the graphs of $y^2 - x = 4$ and $y = x$.



thin rect. has area $y - (y^2-4)\,dy = [y - y^2 + 4]\,dy$

total area $= \int_a^b [y - y^2 + 4]\,dy = \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 + 4y\right)\Big|_a^b$

$= \left(\frac{1}{2}b^2 - \frac{1}{3}b^3 + 4b\right) - \left(\frac{1}{2}a^2 - \frac{1}{3}a^3 + 4a\right)$

with $b = \frac{1+\sqrt{17}}{2}$, $a = \frac{1-\sqrt{17}}{2}$

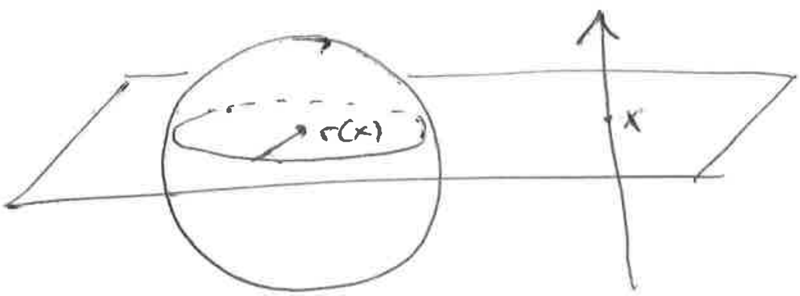
to find a, b solve

$y^2 - 4 = y$

$y^2 - y - 4 = 0$

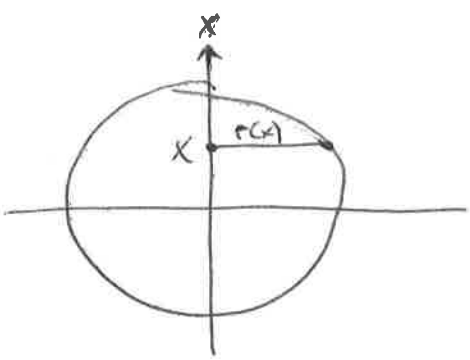
$y = \frac{1 \pm \sqrt{17}}{2}$

2. Prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.



area of cross section

$= \pi r(x)^2 = \pi(r^2 - x^2)$

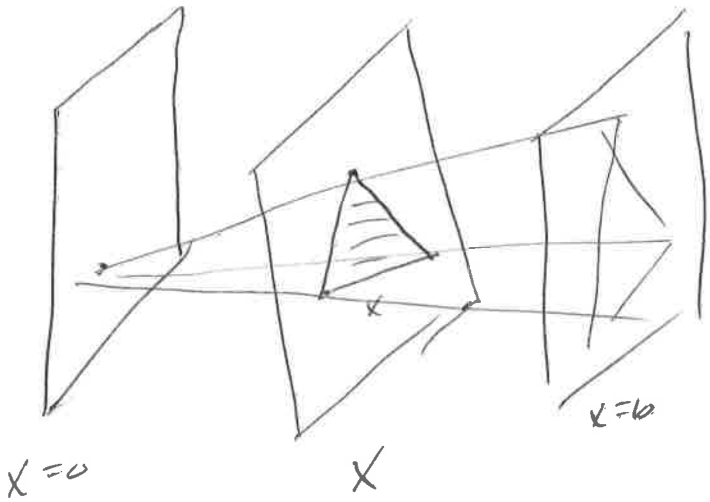


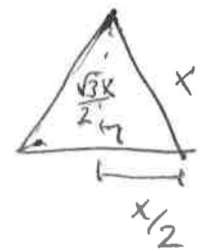
$r(x) = \sqrt{r^2 - x^2}$

total volume $= 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\left(r^2x - \frac{1}{3}x^3\right)\Big|_0^r$

$= 2\pi\left(\frac{2}{3}r^3 - 0\right) = \frac{4\pi}{3}r^3$

3. A solid is contained between the planes $x = 0$ and $x = 10$. When the solid is sliced by the plane perpendicular to the x-axis with x-coordinate x, the resulting cross-section is an equilateral triangle with sides of length x. Find the volume of the solid.



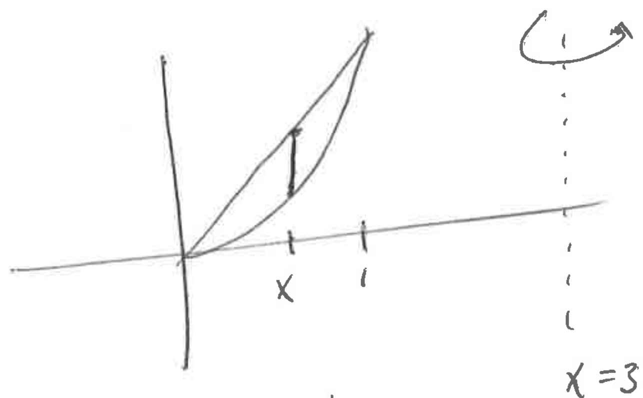


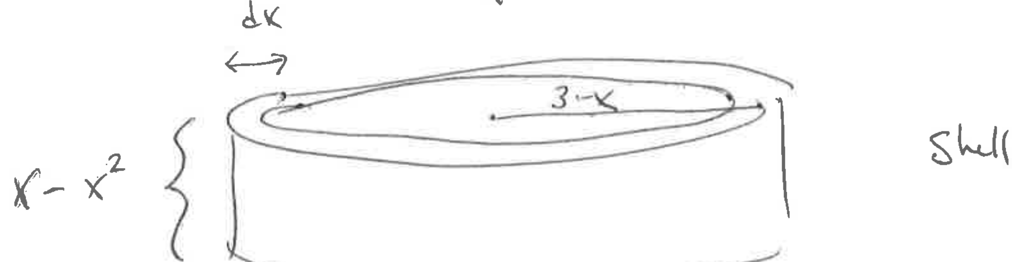
$$A(x) = \frac{1}{2}\cdot x \cdot \frac{\sqrt{3}}{2}x$$

$$= \frac{\sqrt{3}}{4}x^2$$

$$\text{total volume} = \int_0^{10} \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{12}\cdot 10^3$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$, $y = x$ about the line $x = 3$.

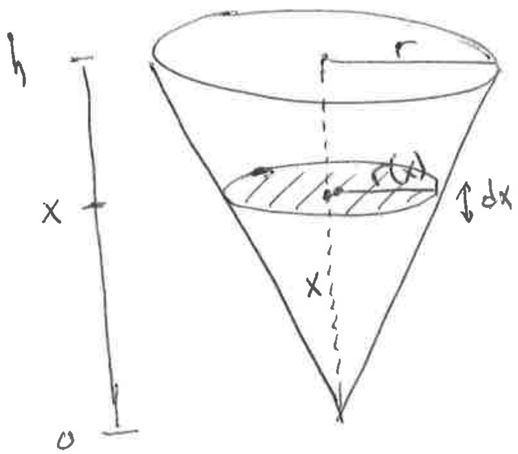




$$\text{vol. of shell} = 2\pi(3-x)(x-x^2)\,dx$$

$$\text{total volume} = \int_0^1 2\pi(3-x)(x-x^2)\,dx = \cdots$$ (multiply out then integrate)

5. A tank shaped like an upside down cone is filled with water. The height of the cone is h and the radius of the base is r. The density of water is 1000 kg per cubic meter. Water is pumped out over the top. How much work is required to empty the tank?



$$\frac{r(x)}{x} = \frac{r}{h} \quad \text{by similarity}$$

$$r(x) = \frac{r}{h} x$$

$$\text{mass of disc} = 1000 \cdot \pi r(x)^2 \, dx$$

$$\text{work to move disc to top} = (h-x) g \, 1000 \pi \, r(x)^2 dx$$

$$\text{total work} = \int_0^h (h-x) g \, 1000\pi \frac{r^2}{h^2} x^2 dx = \cdots$$

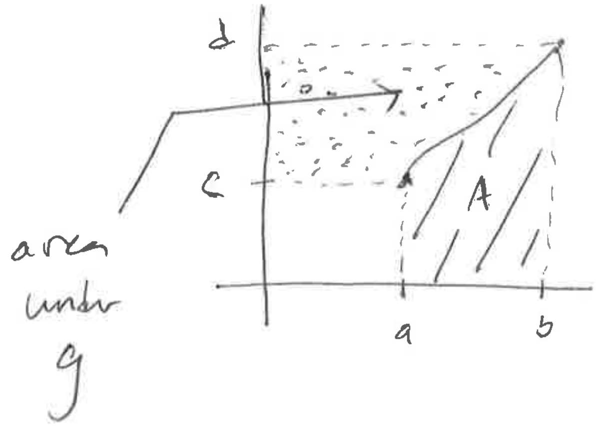
6. Let $p(t)$ denote the position of a particle as a function of time t, for $a \leq t \leq b$. Show that the average velocity of the particle on the interval $[a, b]$ is equal to the average value of the velocity function $v(t) = p'(t)$ on the interval.

$$\text{ave. vel. on } [a,b] = \frac{p(b) - p(a)}{b-a}$$

$$\text{ave value of vel. function} = \frac{1}{b-a} \int_a^b p'(t) \, dt$$

$$= \frac{1}{b-a} \left(p(b) - p(a) \right) \quad (\text{by FTC})$$

7. Let $f : [a, b] \to [c, d]$ be a one-to-one and onto function. Let $g : [c, d] \to [a, b]$ denote the inverse. Assume $0 \le a < b$ and $0 \le c < d$. The area under the graph of f is A. Find the area under the graph of g.



$$A + (\text{area under } g) = a(d-c) + (d-c)(b-a) + c(b-a) = a(d-c) + d(b-a)$$

So $\quad$ area under $g = a(d-c) + d(b-a) - A$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has an inverse f^{-1}. Does f^2 have an inverse? Why or why not? If so, what is the inverse? Answer the same questions for f^3.

f^2 does not have inverse:

Since range of f is $\mathbb{R}$, there is some a with $f(a) = -1$ and some b with $f(b) = +1$. Then $f^2(a) = (-1)^2 = 1 = (1)^2 = f^2(b)$, so f^2 is not one-to-one and hence not invertible.

f^3 has an inverse:

Let $g(x) = f^{-1}(x^{1/3})$. Then

$f^3 \circ g(x) = (f(f^{-1}(x^{1/3})))^3 = (x^{1/3})^3 = x$ and

$g \circ f^3(x) = f^{-1}(((f(x))^3)^{1/3}) = f^{-1}(f(x)) = x$.

So g is the inverse of f^3.

9. Evaluate the following integrals

a) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C$

$u = \cos x$, $du = -\sin x \, dx$ $\quad = -\ln|\cos x| + C$

$= \ln|\sec x| + C$

b) $\int \frac{\sin x}{e^{\cos x}} dx = -\int \frac{du}{e^u} = -\int e^{-u} du$

$u = \cos x$, $du = -\sin x \, dx$ $\quad = e^{-u} + C = e^{-\cos x} + C$

c) $\int (e^u + e^{-u})^2 \, du = \int \left(e^{2u} + 2 + e^{-2u}\right) du = \frac{1}{2} e^{2u} + 2u - \frac{1}{2} e^{-2u} + C$

d) $\int \frac{x}{-x^2+1} dx = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln|u| + C$

$u = -x^2 + 1$

$du = -2x \, dx$ $\quad = -\frac{1}{2} \ln|-x^2+1| + C$

e) $\int \frac{t^2}{t+2} dt = \int \frac{(u-2)^2}{u} du = \int \frac{u^2-4u+4}{u} du = \int \left(u - 4 + \frac{4}{u}\right) du$

$u = t+2$, $du = dt$, $t = u-2$

$= \frac{1}{2} u^2 - 4u + 4 \ln|u| + C$

$= \frac{1}{2}(t+2)^2 - 4(t+2) + 4 \ln|t+2| + C$

f) $\int 2^x \cdot 3^x \, dt$

$= \int e^{x \cdot \ln 2} e^{x \ln 3} dx = \int e^{x(\ln 2 + \ln 3)} dx$

$= \frac{1}{\ln 2 + \ln 3} e^{x(\ln 2 + \ln 3)} + C$

10. Find the derivatives of the following functions

a) $(x^2 2^x)' = 2x\ 2^x + x^2 \cdot \ln 2 \cdot 2^x$

b) $e^x e^x = e^{2x}$, so $\frac{d}{dx}(e^x e^x) = 2e^{2x}$

c) $x^{\cos x} = y$, $\ln y = \cos x \cdot \ln x$

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln x + \frac{\cos x}{x}$, $\frac{dy}{dx} = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$

d) $\frac{x^{3/4}(x-1)^2}{(\cos x)^3} = y$, $\ln y = \frac{3}{4}\ln x + 2\ln(x-1) - 3\ln\cos x$

so $\frac{dy}{dx} = y\left(\frac{3}{4}\cdot\frac{1}{x} + \frac{2}{x-1} + 3\frac{\sin x}{\cos x}\right) = \cdots$

e) $(x\ln|x| - x)'$

$= 1 \cdot \ln|x| + x \cdot \frac{1}{x} - 1 = \ln|x|$

f) $\ln\left(x^2(\cos x)\sqrt{x+1}\right) = 2\ln x + \ln\cos x + \frac{1}{2}\ln(x+1)$

So $\frac{d}{dx}(\cdots) = \frac{2}{x} - \frac{\sin x}{\cos x} + \frac{1}{2}\cdot\frac{1}{x+1}$

g) $\log_2(5x^2+1) = \frac{\ln(5x^2+1)}{\ln 2}$

So $\frac{d}{dx}(\cdots) = \frac{1}{\ln 2} \cdot 10x \cdot \frac{1}{5x^2+1}$

h) $\ln|2x + \tan x|$

$\frac{d}{dx}(\cdots) = \frac{2 + \sec^2 x}{2x + \tan x}$

i) $(\ln x)^3$

$\frac{d}{dx}(\cdots) = 3(\ln x)^2 \cdot \frac{1}{x}$

11. Recall that ln is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

a) Show that ln is an increasing function.

b) Show that ln is concave down.

c) Explain why ln is a continuous function.

a) $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$, so $\ln x$ is increasing

b) $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2} < 0$, so $\ln x$ is concave down

c) The function $\int_1^x \frac{1}{t}\,dt$ is continuous (in the x variable) because the integrand $\frac{1}{t}$ is continuous. (This is a theorem.)