Math 2924-050
Spring 2015
Exam 3

Name:____

Problem	**Points**
Problem 1 (20 pts)	
Problem 2 (20 pts)	
Problem 3 (20 pts)	
Problem 4 (20 pts)	
Problem 5 (20 pts)	
Total	

Instructions:

- You are allowed to use a calculator and one 4 inch by 6 inch index card of formulas.
- You must show your work on any problem that requires a solution of more than one or two lines.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (20 points) Find the Taylor series (centered at 0) of the following functions.

a) $x\sin(x^2)$

$$= x \sum_{n=0}^{\infty} \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(2n+1)!}$$

b) $\ln(1-2x)$ $$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-2x)^n}{n} = \sum_{n=1}^{\infty} -\frac{2^n}{n} x^n$$

c) $\int e^{x^2}\, dx$ $$= \int \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\, dx = \sum_{n=0}^{\infty} \left(\int \frac{x^{2n}}{n!}\, dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C$$

d) $\frac{1}{(1-x)^2}$ $$= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\frac{d}{dx} x^n\right)$$

$$= \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} (n+1) x^n$$

2. (20 points) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n}(x-1)^n.$$

a) Find the radius of convergence and the interval of convergence.

use ratio test: $\lim_{n\to\infty} \dfrac{\frac{2^{n+1}}{n+1}|x-1|^{n+1}}{\frac{2^n}{n}|x-1|^n} = 2|x-1| \lim_{n\to\infty} \dfrac{n}{n+1}$

$= 2|x-1| < 1$ when $|x-1| < 1/2$

So radius of conv. $= 1/2$

At endpoint $x = 1 + 1/2 = 3/2$: $\sum_{n=0}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum \frac{1}{n}$ div.

At endpoint $x = 1 - 1/2 = 1/2$: $\sum_{n=0}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n = \sum (-1)^n \frac{1}{n}$ conv.

b) Find $f^{(100)}(1)$.

So interval of conv. is $[1/2, 3/2)$

$$\frac{f^{(100)}(1)}{100!} = \text{coefficient of } (x-1)^{100} = \frac{2^{100}}{100}$$

So $f^{(100)}(1) = \dfrac{100!}{100} \cdot 2^{100} = 99! \cdot 2^{100}$

3. (20 points) Test the following series for convergence/divergence:

$$\sum_{n=1}^{\infty}\left(\frac{e^{1/n}-1-\frac{1}{n}}{1/n^2}\right)^n.$$

use root test: $\lim\limits_{n\to\infty}\left[\left(\frac{e^{1/n}-1-1/n}{1/n^2}\right)^n\right]^{1/n} = \lim\limits_{n\to\infty}\frac{e^{1/n}-1-1/n}{1/n^2}$

$$= \lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2} < 1$$

$\left(\begin{array}{l}\text{let } x = 1/n \\ x\to 0 \text{ as } n\to\infty\end{array}\right)$ So conv. by root test

4. (20 points) Consider the parametric curve defined by

$$x = 2\cosh t, \qquad y = 3\sinh t.$$

a) Find all points on the curve where the tangent line is horizontal or vertical.

$$\frac{dx}{dt} = 2\sinh t = 0 \quad \text{when} \quad t = 0$$

$$\frac{dy}{dt} = 3\cosh t = 0 \quad \text{has no solutions}$$

So vert. tang. at $t=0$, $(x,y) = (2, 0)$

no ~~vert~~ horiz. tangent

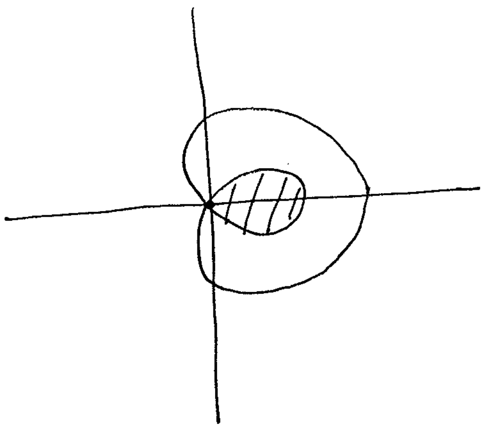
b) Find a cartesian equation for the curve by eliminating t.

$$\frac{x}{2} = \cosh t, \quad \frac{y}{3} = \sinh t$$

$$1 = \cosh^2 t - \sinh^2 t = \frac{x^2}{4} - \frac{y^2}{9}$$

So $$1 = \frac{x^2}{4} - \frac{y^2}{9} \quad \text{(hyperbola)}$$

5. (20 points) Below is a sketch of the curve defined in polar coordinates by $r = 1 + 2\cos\theta$. Write down the integral that computes the area of the shaded region. You do not need to evaluate the integral.

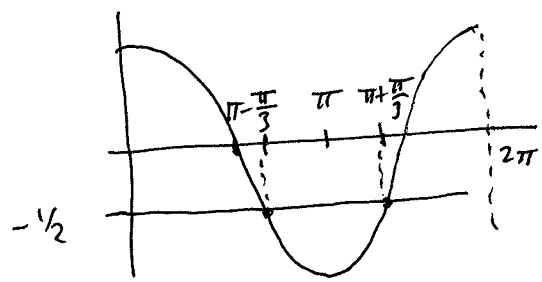


solve $0 = 1 + 2\cos\theta$

to get self intersection point,

i.e. $\cos\theta = -1/2$

there are 2 solutions in interval $0 \le \theta \le 2\pi$



So area $= \displaystyle\int_{\pi-\pi/3}^{\pi+\pi/3} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} (1 + 2\cos\theta)^2 \, d\theta$