# Math 2924-050
# Spring 2015
# Exam 1

Name: SOLUTIONS

| Problem | Points |
|---|---|
| Problem 1 (15 pts) | |
| Problem 2 (15 pts) | |
| Problem 3 (15 pts) | |
| Problem 4 (20 pts) | |
| Problem 5 (15 pts) | |
| Problem 6 (20 pts) | |
| Total | |

**Instructions:**

- You are allowed to use a calculator and one 3 inch by 5 inch index card of formulas.
- A list of integral formulas is provided on the last page of the exam.
- You must show your work on any problem that requires a solution of more than one or two lines.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (15 points) $f(x)$ and $f^{-1}(x)$ are inverse functions of each other which satisfy

$$f(0) = 2, \quad f(2) = 3, \quad f(3) = 10, \quad f'(3) = 5, \quad (f^{-1})'(2) = 1/3.$$

Find the following:

a) $f^{-1}(3)$

$f^{-1}(3) = 2$

b) $(f^{-1})'(10)$

$$(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(3)} = \frac{1}{5}$$

c) $f'(0)$

$$f'(0) = \frac{1}{(f^{-1})'(f(0))} = \frac{1}{(f^{-1})'(2)} = \frac{1}{\frac{1}{3}} = 3$$

2. (15 points) Find the limit

$$\lim_{x\to 0+} \left[\csc x - x^{-1}\right].$$

$$= \lim_{x\to 0^+} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x\to 0^+} \frac{x - \sin x}{x \sin x} \quad \left( \text{Type } \frac{0}{0} \text{ use L'H.} \right)$$

$$= \lim_{x\to 0^+} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{\sin 0}{\cos 0 + \cos 0 - 0} = 0$$

$\left( \text{Type } \frac{0}{0}, \text{ use L'H} \right)$

3. (15 points) Find the derivatives of the following functions:

a) $\ln(\sec^{-1} x)$

$$\frac{1}{\sec^{-1} x} \cdot \frac{1}{x\sqrt{x^2-1}}$$

b) $1^x 2^x 3^x 4^x 5^x$ $= (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)^x = 120^x$

$$\ln 120 \cdot 120^x$$

c) $\sinh(ax)\cosh(bx)$ ($a$ and $b$ are constants)

$$a \cosh(ax) \cosh(bx) + b \sinh(ax) \cosh(bx)$$

4. (20 points) Find

$$\int \frac{2x^2+4x+5}{x^3+4x^2+5x}\,dx$$

$$\frac{2x^2+4x+5}{x(x^2+4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+5}$$

$$2x^2+4x+5 = A(x^2+4x+5) + (Bx+C)x$$

plug in $x=0$ : $\quad 5 = 5A \Rightarrow A = 1$

$x^2$ coeff : $\quad 2 = A + B \Rightarrow B = 1$

$x$ coeff : $\quad 4 = 4A + C \Rightarrow C = 0$

So $$\int \frac{2x^2+4x+5}{x^3+4x^2+5x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4x+5}\,dx$$

$$= \ln|x| + \int \frac{x}{(x+2)^2+1}\,dx = \ln|x| + \frac{1}{2}\int \frac{2(x+2)}{(x+2)^2+1}\,dx - \int \frac{2}{(x+2)^2+1}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln\left|(x+2)^2+1\right| - 2\tan^{-1}(x+2) + C$$

5. (15 points) Use integration by parts to prove the formula

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

let $u = (\ln x)^n$, $du = \frac{n (\ln x)^{n-1}}{x} dx$

$dv = dx$, $v = x$

$$= uv - \int v \, du$$

$$= x (\ln x)^n - \int x \cdot \frac{n (\ln x)^{n-1}}{x} dx$$

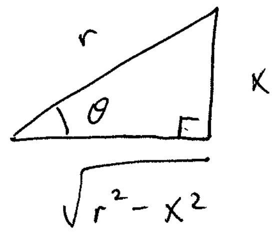
$$= x (\ln x)^n - n \int (\ln x)^{n-1} dx$$

6. (20 points) Find

$$\int \sqrt{r^2 - x^2}\,\mathrm{d}x$$

let $x = r\sin\theta$

$dx = r\cos\theta\, d\theta$




$$\int \sqrt{r^2-x^2}\,dx = \int \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta = \int r\cos\theta \cdot r\cos\theta\, d\theta$$

$$= r^2 \int \cos^2\theta\, d\theta = r^2 \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{r^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{r^2}{2}\left(\theta + \sin\theta\cos\theta\right) + C = \frac{r^2}{2}\left(\sin^{-1}\frac{x}{r} + \frac{x}{r}\cdot\frac{\sqrt{r^2-x^2}}{r}\right) + C$$

$$= \frac{r^2}{2}\cdot\sin^{-1}\frac{x}{r} + \frac{x}{2}\cdot\sqrt{r^2-x^2} + C$$

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

**Table of Integration Formulas** Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

*19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$