How to integrate $\int \sec^m x \tan^n x \, dx$

with $m, n \geq 0$ integers.

I. $m \geq 2$ even

- break off $\sec^2 x$
- replace other powers of $\sec^2 x$ with $1 + \tan^2 x$
- let $u = \tan x$

Example

For $k \geq 1$, $\int \sec^{2k} x \tan^n x \, dx = \int \sec^{2k-2} x \tan^n x \sec^2 x \, dx$

$$= \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx = \int (1 + u^2)^{k-1} u^n \, du = \cdots$$

II. $m \geq 1$, $n \geq 1$ odd

- break off $\sec x \tan x$
- replace other powers of $\tan^2 x$ with $\sec^2 x - 1$
- let $u = \sec x$

Example

For $k \geq 0$, $m \geq 1$, $\int \sec^m x \tan^{2k+1} x \, dx = \int \sec^{m-1} x \tan^{2k} x \sec x \tan x \, dx$

$$= \int \sec^{m-1} x \, (\sec^2 x - 1)^k \sec x \tan x \, dx$$

$$= \int u^{m-1} (u^2 - 1)^k \, du = \cdots$$

III. $\underline{m = 0}$

- $\displaystyle\int \tan x \, dx = \ln |\sec x| + C$

- For $n \geq 2$, $\displaystyle\int \tan^n x \, dx = \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= (\text{Type I}) + (\text{Type III with smaller } n)$$

IV. $\underline{m \text{ odd}, \; n = 0}$

- $\displaystyle\int \sec x \, dx = \ln |\sec x + \tan x| + C$

- For $m \geq 3$, integrate by parts with $dv = \sec^2 x \, dx$, $u = \sec^{m-2} x \, dx$

$$\int \sec^m x \, dx = \sec^{m-2} x \tan x - (m-2) \int \sec^{m-2} x \tan^2 x \, dx$$

$$= \sec^{m-2} x \tan x - (m-2) \int \sec^{m-2} x \, (\sec^2 x - 1) \, dx$$

$$= \sec^{m-2} x \tan x - (m-2) \int \sec^m x + (m-2) \int \sec^{m-2} x \, dx$$

$$\Rightarrow \int \sec^m x \, dx = \frac{1}{m-1} \sec^{m-2} x \tan x + \underbrace{\frac{m-2}{m-1} \int \sec^{m-2} x \, dx}_{(\text{Type IV with smaller } m)}$$

V. $\underline{m \text{ odd}, \; n \text{ even}}$

- Replace powers of $\tan^2 x$ with $\sec^2 x - 1$:

$$\int \sec^m x \tan^{2k} x \, dx = \int \sec^m x \, (\sec^2 x - 1)^k \, dx = (\text{Sum of Type IV's})$$