Test for conv. / div.

1) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

Alt. series, $|a_n| = \frac{n}{n^2+2}$

i) let $f(x) = \frac{x}{x^2+2}$, so $f'(x) = \frac{1\cdot(x^2+2) - x\cdot(2x)}{(x^2+2)^2} = \frac{-x^2+2}{(x^2+2)^2} < 0$ for $x \geq 2$

So $|a_{n+1}| \leq |a_n|$ for $n \geq 2$

ii) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{1}{n+2/n} = 0$, so AST $\Rightarrow$ converges

2) $\sum_{K=1}^{\infty} \frac{2^K}{K!}$

Try ratio test: $a_n = \frac{2^n}{n!}$, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

So conv. by ratio test.

3) $\sum_{K=1}^{\infty} \frac{1}{K\sqrt{K^2+1}}$

Compare to $\sum \frac{1}{K^2}$ using comparison test (or limit comp. test)

$$\frac{1}{K\sqrt{K^2+1}} \leq \frac{1}{K\sqrt{K^2}} = \frac{1}{K^2}$$

So $\sum \frac{1}{K\sqrt{K^2+1}} \leq \sum \frac{1}{K^2} < \infty$ (p-series with $p=2$)

So $\sum \frac{1}{K\sqrt{K^2+1}}$ conv.

4) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{e^n}$

Test for abs. conv.: $\left|\frac{\sin n}{e^n}\right| = \frac{|\sin n|}{e^n} \leq \frac{1}{e^n}$

and $\sum \frac{1}{e^n}$ conv. (geo. series with $r = \frac{1}{e}$, $|r|<1$)

So $\sum \frac{\sin n}{e^n}$ conv. abs., and hence conv.

5) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$

Compare to $\sum \frac{1}{n}$ using limit comp test:

$$\lim_{n\to\infty} \frac{\sqrt[n]{2} - 1}{1/n} = \lim_{n\to\infty} \frac{-1/n^2 \cdot \ln 2 \cdot \sqrt[n]{2}}{-1/n^2} = \lim_{n\to\infty} \ln 2 \cdot 2^{1/n} = \ln 2$$

(Type $\frac{0}{0}$, use L'H.)

So $\sum (\sqrt[n]{2} - 1)$ div. b/c $\sum 1/n$ div.

by limit comp. test

6) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$

Use root test:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\left|\frac{n-1}{n}\right|^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n$$

$$= \lim_{n\to\infty} (1 - 1/n)^n = e^{-1} < 1,$$ so conv. by root test.