Review Problems for Exam III

1) Test for conv./div.

a) $$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$$

alternating and decreasing, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$, so conv.

by AST

b) $$\sum_{K=1}^{\infty} \frac{K \ln K}{(K+1)^3}$$

For large K, $\ln K < K^{1/2}$, so $\frac{K \ln K}{(K+1)^3} \le \frac{K \cdot K^{1/2}}{(K+1)^3} = \frac{K^{3/2}}{(K+1)^3}$

$\le \frac{K^{3/2}}{K^3} = \frac{1}{K^{3/2}}$, and $\sum \frac{1}{K^{3/2}}$ conv. b/c p-series with $p = 3/2 > 1$,

so conv. by comparison test

c) $$\sum_{n=1}^{\infty} \frac{n^{4n}}{(n!)^n}$$

apply root test: $\lim_{n\to\infty} \left(\frac{n^{4n}}{(n!)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^4}{n!} = 0$,

So conv. by root test

2. Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n$.

a) Find the interval of convergence.

Use ratio test: $\lim_{n\to\infty} \dfrac{|x|^{n+1}/((n+1)!)^2}{|x|^n/(n!)^2} = \lim_{n\to\infty} |x| \cdot \frac{n!}{(n+1)!} \cdot \frac{n!}{(n+1)!}$

$= |x| \cdot \lim_{n\to\infty} \frac{1}{(n+1)} \cdot \frac{1}{(n+1)} = 0$ for all x. So interval of conv. is

$(-\infty, \infty)$.

b) Find $f'(x)$.

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{1}{(n!)^2} x^n = \sum_{n=0}^{\infty} \frac{n}{(n!)^2} x^{n-1}.$$

c) Find $\lim_{x\to 0} \dfrac{f(x) - (1 + x + \frac{1}{4}x^2)}{x^3}$

$$f(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{576}x^4 + \dots$$

~~all these terms have x^4 or higher~~

So $\lim_{x\to 0} \dfrac{f(x) - (1 + x + \frac{1}{4}x^2)}{x^3} = \cancel{\lim_{x\to 0} \dfrac{\frac{1}{36}x^3 + \dots}{x^3} = \frac{1}{36}}$

$$= \lim_{x\to 0} \frac{\frac{1}{36}x^3 + \frac{1}{576}x^4 + \dots}{x^3} = \lim_{x\to 0} \left(\frac{1}{36} + \frac{1}{576}x + \dots\right) = \frac{1}{36} + 0 + 0 + 0 + \dots$$

$$= \frac{1}{36}$$

3. Find the Taylor series for e^x centered at $a=-1$.

$f(x)=e^x$, $f^{(n)}(x)=e^x$, $f^{(n)}(-1)=e^{-1}$

So $$e^x=\sum_{n=0}^{\infty}\frac{e^{-1}}{n!}(x+1)^n$$

4. Find the Taylor series (centered at $a=0$) for the following functions:

a) $$e^{2x}=\sum_{n=0}^{\infty}\frac{1}{n!}(2x)^n=\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

b) $$x\tan^{-1}\left(\frac{x}{2}\right)=x\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{x}{2}\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{2n+1}}\cdot x^{2n+2}$$

c) $$\frac{x^3}{1+x^2}=x^3\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n+3}$$

d) $$2x\sinh x+\cos 3x=2x\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}+\sum_{n=0}^{\infty}\frac{(3x)^{2n}}{(2n)!}$$

$$=\sum_{n=0}^{\infty}\frac{2}{(2n+1)!}x^{2n+2}+\sum_{n=0}^{\infty}\frac{3^{2n}}{(2n)!}x^{2n}=1+\sum_{n=1}^{\infty}\left[\frac{3^{2n}}{(2n)!}+\frac{2}{(2n-1)!}\right]x^{2n}$$

e) $$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

5. Use a power series to evaluate $\int e^{-x^2}\,dx$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

So $$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\,dx = \sum_{n=0}^{\infty} \left[\int \frac{(-1)^n}{n!} x^{2n}\,dx\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$$

6. It can be shown (using methods of Calc 3) that

$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Use this fact and your answer for

#5 to find a series representation of $\sqrt{\pi}$.

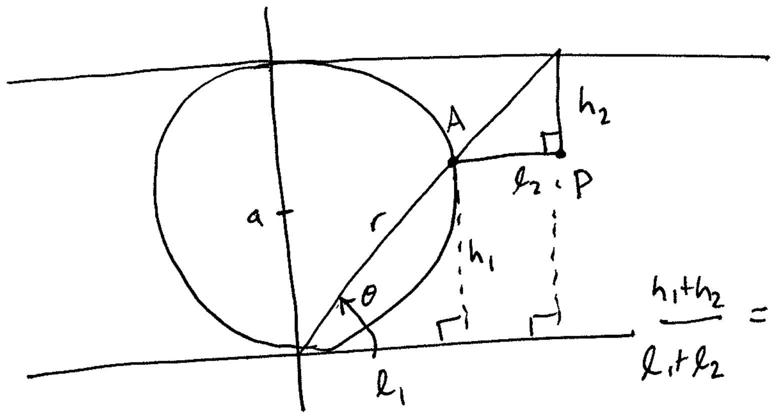
Let $$F(x) = \int_0^x e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$$ ($C = 0$ b/c $F(0) = 0$)

Then $$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx = 2\lim_{x\to\infty} F(x)$$

$$= 2\lim_{x\to\infty}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}\right)$$

(Sorry... this is not quite a power series representation)

7. Find parametric equations, with θ as the parameter, for the curve consisting of all ~~points~~ possible positions of the point P in the figure.



$$h_1 + h_2 = 2a$$

$$\frac{h_1}{l_1} = \tan\theta$$

$$\frac{h_1+h_2}{l_1+l_2} = \frac{h_1}{l_1} = \frac{h_2}{l_2} \quad \text{(by similar triangles)}$$

$$\Downarrow$$

$$\frac{2a}{l_1+l_2} = \tan\theta$$

~~tan~~. $l_1 + l_2 = 2a\cot\theta$

$$\underbrace{(r\cos\theta)^2 + (r\sin\theta - a)^2 = a^2}_{} \quad \text{(since } A \text{ lies on circle)}$$

$$\Downarrow$$

$$r = 2a\sin\theta$$

$$h_1 = r\sin\theta = 2a\sin^2\theta$$

So $\boxed{P = (2a\cot\theta,\ 2a\sin^2\theta)}$

8. Find the arc length of the curve ~~$x = e^t + e^{-t}$, $y = 5 - 2t$~~,

$x = t\sin t$, $y = t\cos t$ $\quad 0 \le t \le 1$.

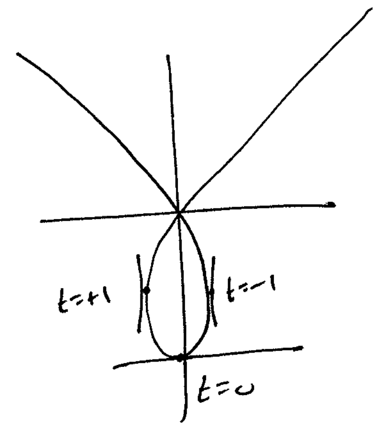
$$\text{Arc length} = \int_0^1 \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt = \int_0^1 \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\, dt$$

$$= \int_0^1 \sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t}\, dt = \int_0^1 \sqrt{1+t^2}\, dt$$

$$\underset{\text{(look up in table of integrals)}}{=} \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right]_0^1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)$$

9. Find the points on the curve where the tangent line is horizontal or vertical. Sketch the curve.

$x = t^3 - 3t,\ y = t^2 - 3.$



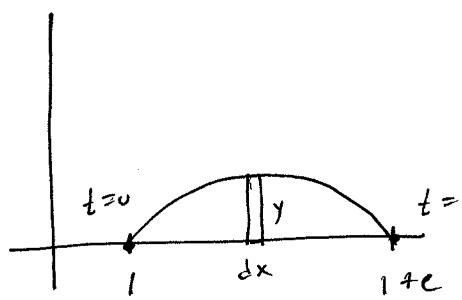
$\frac{dx}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$ ← vert. tang. lines

$\frac{dy}{dt} = 2t = 0$ when $t = 0$ ← horiz. tang. lines

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

10. Find the area enclosed by the x-axis and the curve

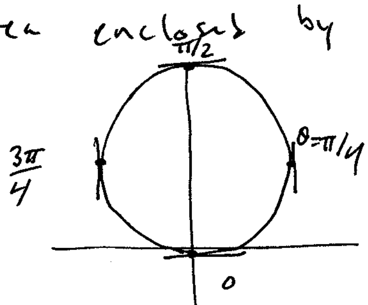
$x = 1 + e^t, \quad y = t - t^2.$



$y = t - t^2 = 0$ when $t = 0, 1$

$\text{Area} = \int_{t=0}^{t=1} y\,dx = \int_0^1 (t - t^2)e^t\,dt = \ldots$

11. Graph the curve $r = 5\sin\theta$ (in polar coordinates). Find points where tangent line is horizontal or vertical. Find area enclosed by the curve.



$x = r\cos\theta = 5\sin\theta\cos\theta = \frac{5}{2}\sin 2\theta = 0$

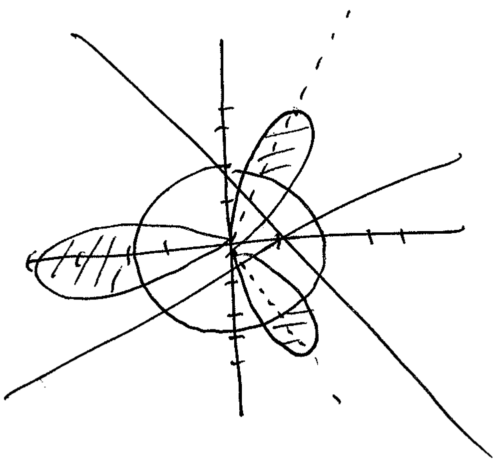
$y = r\sin\theta = 5\sin^2\theta$

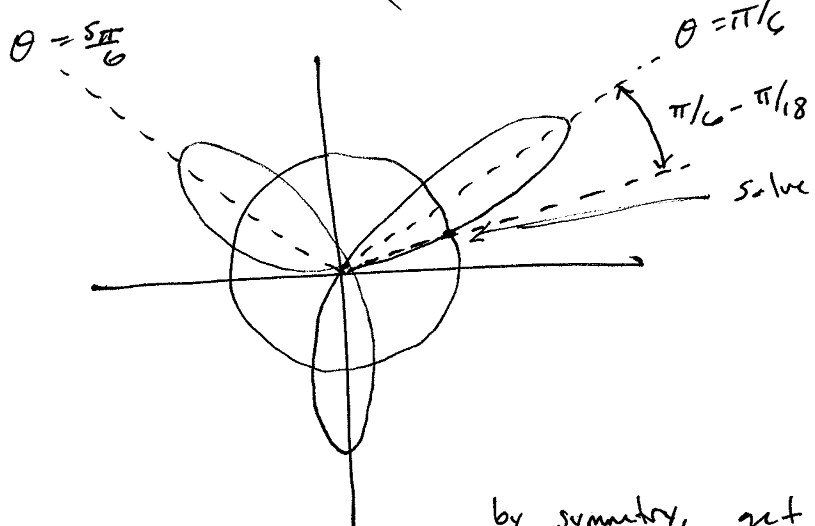
$\frac{dx}{dt} = 5\cos 2\theta = 0$ when $\theta = \cancel{\pi/2}\ \pi/4,\ 3\pi/4$ vert. tang. lines

$\frac{dy}{dt} = 10\sin\theta\cos\theta = \frac{5}{2}\sin 2\theta = 0$ when $\theta = 0, \frac{\pi}{2}$ horiz. tang. lines

$\text{Area} = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} \frac{1}{2} 25\sin^2\theta = \frac{25}{2}\int_0^{\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{25}{2}\cdot\frac{\pi}{2} = \frac{25}{4}\pi$

12. Find the points of intersection of the curves $r = 2$ and $r = 4\sin(3\theta)$. Find the area of the region outside the circle $r=2$ and inside $r = 4\sin(3\theta)$.





$2 = 4\sin(3\theta)$

$\Rightarrow 3\theta = \pi/6$

$\theta = \pi/18$

by symmetry, get intersection points are

$\left(2, \pi/18\right), \left(2, \pi/6 + \pi/9\right), \left(2, \frac{5\pi}{6} \pm \pi/9\right)$

$\left(2, \frac{3\pi}{2} \pm \pi/9\right)$

$$\text{Area} = 6\int_{\pi/18}^{\pi/6} \frac{1}{2}\left(16\sin^2(3\theta) - 4\right)d\theta = 48\int_{\pi/18}^{\pi/6} \sin^2 3\theta\, d\theta - 12\left(\frac{\pi}{6} - \frac{\pi}{18}\right)$$

$$= 24\int_{\pi/18}^{\pi/6}(1 - \cos 6\theta)\,d\theta - \frac{12\pi}{9} = 24\left(\frac{\pi}{6} - \frac{\pi}{18}\right) - 6\sin 6\theta\Big|_{\pi/18}^{\pi/6} - \frac{12\pi}{9}$$

$$= 6\sin\pi/3 = 6\,\frac{\sqrt{3}}{2} = 3\sqrt{3}$$