Growth Rates and Convergence of Series

Recall:

$$\ln n \ll \underset{(p>0)}{n^p} \ll r^n \ll n! \ll n^n$$

Today's Goal: Prove $\sum \dfrac{(\text{slower term})}{(\text{faster term})}$ converges

(if (faster term) $= n^p$ this is true only if $p > 1$)

1) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ conv. if $p > 1$.

Integral Test: $\dfrac{\ln n}{n^p} \geq 0$ for $n \geq 1$, and $\dfrac{d}{dx}\left(\dfrac{\ln x}{x^p}\right) = \dfrac{\frac{1}{x}\cdot x^p - \ln x \cdot p x^{p-1}}{(x^p)^2} = \dfrac{x^{p-1}(1-p\ln x)}{(x^p)^2} \leq 0$

for x large enough. Thus integral test applies.

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{R\to\infty} \int_1^R \frac{\ln x}{x^p}\,dx = \lim_{R\to\infty} \left[\frac{1}{-p+1}x^{1-p}\ln x - \int \frac{1}{-p+1}x^{-p}\,dx\right]\Bigg|_1^R = \lim_{R\to\infty}\left[\frac{1}{-p+1}x^{1-p}\ln x + \frac{1}{p-1}\cdot\frac{1}{-p+1}x^{-p+1}\right]\Bigg|_1^R$$

$u = \ln x$, $du = \frac{dx}{x}$, $dv = x^{-p}dx$, $v = \frac{1}{-p+1}x^{-p+1}$

$= \dfrac{-1}{p-1}\cdot\dfrac{1}{-p+1}$ b/c $p > 1$. So converges.

2) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n^p}{r^n}$ conv. if $p > 0$, $r > 1$.

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{(n+1)^p}{r^{n+1}}\cdot\frac{r^n}{n^p} = \frac{1}{r}\lim_{n\to\infty}\frac{(n+1)^p}{n^p} = \frac{1}{r} < 1$

So converges.

3) Prove that $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ conv.

Ratio Test:

$$\lim_{n\to\infty} \frac{r^{n+1}}{(n+1)!} \cdot \frac{n!}{r^n} = \lim_{n\to\infty} \frac{r}{n+1} = 0 \text{, so converges.}$$

4) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ conv.

Ratio Test: $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)}{(n+1)^{n+1}} \cdot n^n = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}}$

$$= \lim_{n\to\infty} \frac{1}{\left[\frac{n+1}{n}\right]^n} = \lim_{n\to\infty} \frac{1}{\left[1+\frac{1}{n}\right]^n} = \frac{1}{e} < 1 \text{, so converges.}$$

5) Prove that $\sum_{n=1}^{\infty} \frac{\ln n + n^p + r^n + \boxed{n!}}{\boxed{n^n} - n! - r^n - n^p - \ln n}$ conv.

Terms are positive for n large enough because $n^n \gg n! + r^n + n^p + \ln n$. So can apply limit comparison test. Compare to $\sum \frac{n!}{n^n}$ because of highest order terms on top and bottom.

$$\lim_{n\to\infty} \frac{\frac{\ln n + n^p + r^n + n!}{n^n - n! - r^n - n^p - \ln n}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\frac{\ln n + n^p + r^n + n!}{n!}}{\frac{n^n - n! - r^n - n^p - \ln n}{n^n}}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{\ln n}{n!} + \frac{n^p}{n!} + \frac{r^n}{n!}}{1 - \frac{n!}{n^n} - \frac{r^n}{n^n} - \frac{n^p}{n^p} - \frac{\ln n}{n^n}} = \frac{1+0+0+0}{1-0-0-0-0}.$$

Since $\sum \frac{n!}{n^n}$ conv. by #4, the series converges.