Things that can be done with power series

1) Evaluate integrals

use power series for e^x to evaluate $\int e^{x^2}\,dx$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, \text{ so}$$

$$\int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} + C$$

2) Solve differential equations

Say we want to solve $f'(x) = f(x)$. (secretly we know from Section 6.5 that solution is $f(x) = C_0 e^x$)

Look for a solution of the form $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

a) Show that $C_n = (n+1)C_{n+1}$. What is the solution?

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)c_{n+1} x^n,$$

$f(x) = f'(x)$ $\Rightarrow$ coeffs. of power series are equal

$\Rightarrow C_n = (n+1)C_{n+1}$

So $C_{n+1} = \frac{C_n}{n+1}$. C_0 can be anything, then $C_1 = \frac{C_0}{1}$, $C_2 = \frac{C_1}{2} = \frac{C_0}{2\cdot 1}$, $C_3 = \frac{C_2}{3} = \frac{C_0}{3\cdot 2\cdot 1}, \ldots,$ $C_n = \frac{C_0}{n!}$, so soltn. is $f(x) = \sum_{n=0}^{\infty} \frac{C_0}{n!} x^n = C_0 e^x$

b) If $f(0) = 1$, what is the unique solution?

$f(x) = C_0 e^x$

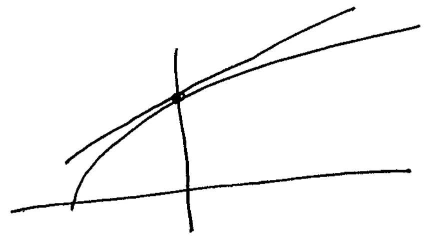
$1 = f(0) = C_0 \Rightarrow f(x) = e^x$ is unique solution

3) Estimate things up to arbitrary order

$$(1+x)^{1/2} = \underbrace{\underbrace{1 + \tfrac{1}{2}x}_{\text{linear}} - \tfrac{1}{8}x^2}_{\text{quadratic}} + \tfrac{3}{48}x^3 - \cdots = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

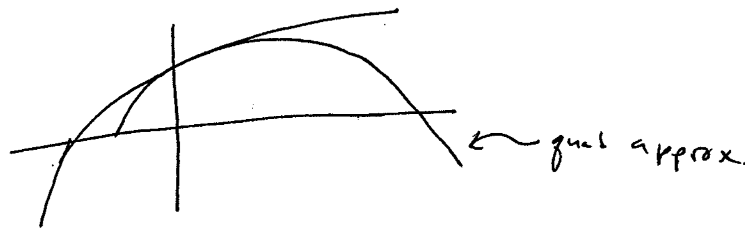
a) Find linear approx. at $x=0$ to $f(x)=\sqrt{1+x}$. Draw a picture.

linear approx. $= 1 + \frac{1}{2}x$



b) Find quadratic approx. at $x=0$ to $f(x)=\sqrt{1+x}$. Draw a picture.

quad. approx. $= 1 + \frac{1}{2}x - \frac{1}{8}x^2$



c) Estimate $\sqrt{1.04}$ without using a calculator. How close is your answer to the exact answer?

$$\sqrt{1.04} = (1+.04)^{1/2} \approx 1 + \frac{.04}{2} - \frac{(.04)^2}{8} = 1 + .02 - \frac{.0016}{8}$$

$$= 1 + .02 - .0002$$

$$= 1.0198$$

Actual answer $= 1.019803903\ldots$

within $.000004$

d.) Doing very little work, calculate $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1/2} - 1 - \frac{1}{2}x + \frac{1}{8}x^2}{x^3 + x^5}$

everything here has x^4 or higher

$$= \lim_{x\to 0} \frac{\frac{3}{48}x^3 + \boxed{\cdots}}{x^3 + x^5} = \lim_{x\to 0} \frac{\frac{3}{48}x^3}{x^3} = \frac{3}{48}$$

since $x \to 0$, lowest order terms are the important ones

4) Analayze growth / decay rates

Use the power series in 3) to explain why

$$\sum_{n=1}^{\infty} \left[\sqrt{1+\frac{1}{n}} - \left(1+\frac{1}{2n}\right) \right]$$ converges.

$$\sqrt{1+\frac{1}{n}} = 1 + \frac{1}{2}\left(\frac{1}{n}\right) - \frac{1}{8}\left(\frac{1}{n}\right)^2 + \frac{3}{48}\left(\frac{1}{n}\right)^3 - \cdots$$ from #3.

So $$\sqrt{1+\frac{1}{n}} - \left(1+\frac{1}{2n}\right) = -\frac{1}{8}\left(\frac{1}{n}\right)^2 + \frac{3}{48}\left(\frac{1}{n}\right)^3 - \cdots$$

$$= -\frac{1}{8n^2}\underbrace{\left[1 - \frac{3}{6}\cdot\frac{1}{n} + \cdots\right]}$$

for large n, this is bounded between 1/2 and 3/2

so $\sqrt{1+\frac{1}{n}} - \left(1+\frac{1}{2n}\right)$ is comparable to $-\frac{1}{8n^2}$, and $\sum -\frac{1}{8n^2}$ conv.

so $\sum \left[\sqrt{1+\frac{1}{n}} - \left(1+\frac{1}{2n}\right)\right]$ conv. also

5) Find derivatives

Let $f(x) = \frac{x^2}{1-x}$. Find power series for $f(x)$, and then find $f^{(100)}(0)$.

$$f(x) = \frac{x^2}{1-x} = x^2 \cdot \frac{1}{1-x} = x^2 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2} = \sum_{n=2}^{\infty} x^n$$

on the other hand, we also know $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

So $\frac{f^{(100)}(0)}{100!} = 1$, hence $f^{(100)}(0) = 100!$

6) Find Sums of Series

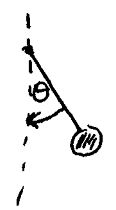
Show that the alternating harmonic series sums to $\ln 2$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

so $\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

7) Simplify equations



The motion of a pendulum solves the equation

$$\theta''(t) = -K \sin \theta(t) \quad \text{for some constant } K > 0.$$

a) If $\theta(t)$ is small, explain why $\sin \theta(t) \approx \theta(t)$.

power series for $\sin\theta$ is $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

if θ is small, the dominant term is θ, so $\sin\theta \approx \theta$.

b) Using a), the equation simplifies to $\theta''(t) = -K\theta(t)$.

Show that $\theta(t) = A\cos\sqrt{K}t + B\sin\sqrt{K}t$ solves this equation for any constants A, B. (This shows that the motion is periodic.)

$$\theta''(t) = (A\cos\sqrt{K}t + B\sin\sqrt{K}t)'' = -KA\cos\sqrt{K}t - KB\sin\sqrt{K}t = -K(A\cos\sqrt{K}t + B\sin\sqrt{K}t)$$
$$= -K\theta(t).$$

8) Euler's Identity

is the remarkable identity relating e, i, and π : ($i = \sqrt{-1}$)

$$e^{i\pi} + 1 = 0$$

Fact: If $\sum_{n=0}^{\infty} c_n x^n$ is a convergent power series, then it makes sense for x to be a complex number.

a) Show that $e^x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n \text{ even}} \frac{x^n}{n!} + \sum_{n \text{ odd}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

b) Show that $\sum_{n=0}^{\infty} \frac{(i\pi)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} = \cos \pi$

$$\sum_{n=0}^{\infty} \frac{(i\pi)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(i^2)^n \pi^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos \pi$$

c) Show that $\sum_{n=0}^{\infty} \frac{(i\pi)^{2n+1}}{\cancel{2n!}\ (2n+1)!} = i \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} = i \sin \pi$

$$\sum_{n=0}^{\infty} \frac{(i\pi)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{i(i^2)^n \pi^{2n+1}}{(2n+1)!} = i \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = i \sin \pi$$

d) Show that $e^{i\pi} + 1 = 0$

$$e^{i\pi} \underset{a)}{=} \sum_{n=0}^{\infty} \frac{(i\pi)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\pi)^{2n+1}}{(2n+1)!} \underset{b) \text{ and } c)}{=} \cos \pi + i \sin \pi = -1 + i \cdot 0 = -1$$

therefore $e^{i\pi} + 1 = 0.$