Growth Rates

Recall

$$\log n \ll n^p \ll r^n \ll n! \ll n^n$$

$$(p>0) \quad (r>1)$$

i.e. $\lim_{n\to\infty} \frac{(\text{faster term})}{(\text{slower term})} = \infty$, $\lim_{n\to\infty} \frac{(\text{slower term})}{(\text{faster term})} = 0$

1) Show that $\lim_{n\to\infty} \frac{n^p}{\log n} = \infty$

$$\lim_{x\to\infty} \frac{x^p}{\log x} = \lim_{x\to\infty} \frac{p x^{p-1}}{\frac{1}{x}} = \lim_{x\to\infty} p x^p = \infty \quad \text{since } p>0$$

(Type $\frac{\infty}{\infty}$, use L'H.)

2) Show that $\lim_{n\to\infty} \frac{r^n}{n^p} = \infty$

$$\lim_{x\to\infty} \frac{r^x}{x^p} = \lim_{x\to\infty} \frac{\ln r \cdot r^x}{p\, x^{p-1}} = \lim_{x\to\infty} \frac{\ln r \cdot \ln r \cdot r^x}{p(p-1)\, x^{p-2}}$$

(Type $\frac{\infty}{\infty}$, use L'H) (Type $\frac{\infty}{\infty}$) (Type $\frac{\infty}{\infty}$)

$$= \cdots = \lim_{x\to\infty} \frac{\ln r \cdot \ln r \cdot \cdots \cdot \ln r \cdot r^x}{(p)(p-1)(p-2)\cdots(p-N)\ \underbrace{x^{p-N}}} = \infty$$

eventually $p-N \leq 0$, no longer Type $\frac{\infty}{\infty}$

3) Show that $\lim_{n\to\infty} \frac{n!}{r^n} = \infty$

Choose integer $N > r$. Let $C = \frac{1\cdot 2\cdot 3\cdot \cdots \cdot N}{r\cdot r\cdot r\cdot \cdots \cdot r}$

Then $\frac{n!}{r^n} = \left(\frac{1\cdot 2\cdot 3\cdot \cdots \cdot N}{r\cdot r\cdot r \cdot \cdots \cdot r}\right)\cdot\left(\frac{(N+1)\cdot \cdots \cdot (n-1)}{r \cdots r}\right)\cdot \frac{n}{r}$

$\geq C \cdot 1 \cdot \frac{n}{r}$

So $\lim_{n\to\infty} \frac{n!}{r^n} \geq \lim_{n\to\infty} \frac{Cn}{r} = \infty$.

4) Show that $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$

$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{1}\right)\cdot\left[\left(\frac{n}{2}\right)\cdots\left(\frac{n}{(n-1)}\right)\left(\frac{n}{n}\right)\right]$

$\geq \lim_{n\to\infty} \frac{n}{1}\cdot 1 = \infty$.