Math 2924-050
Fall 2015
Exam 3

Name:____

Problem	**Points**
Problem 1 (3 pts)	
Problem 2 (3 pts)	
Problem 3 (3 pts)	
Problem 4 (3 pts)	
Problem 5 (3 pts)	
Total (15 pts)	

Instructions:

- You are allowed to use a calculator and one 4 inch by 6 inch index card of formulas.
- A list of integral formulas is provided on the last page of the exam.
- You must show your work on any problem that requires a solution of more than one or two lines.
- Some problems are easier than others. Make sure to attempt all the problems before spending a lot of time on the hard ones.

1. (3 points) Determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

compare to $\sum \frac{1}{n^2}$ using limit comp. test:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n\sqrt{n^2-1}}} = \lim_{n\to\infty} \frac{n\sqrt{n^2-1}}{n^2} = \lim_{n\to\infty} \sqrt{1-\frac{1}{n^2}} = 1$$

since $\sum \frac{1}{n^2}$ conv. (b/c p-series with $p=2>1$),

$\sum \frac{1}{n\sqrt{n^2-1}}$ converges also

2. (3 points) Determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{3^n + 5^n}{4^n + 5^n}$$

$$\lim_{n \to \infty} \frac{3^n + 5^n}{4^n + 5^n} = \lim_{n \to \infty} \frac{\frac{3^n}{5^n} + 1}{\frac{4^n}{5^n} + 1} = \frac{0+1}{0+1} = 1 \neq 0,$$

So diverges by test for divergence

3. (3 points) Find the Taylor series centered at 2 for the function $f(x) = e^{-2x}$. What is the radius of convergence?

$f(x) = e^{-2x}$

$f'(x) = -2 \cdot e^{-2x}$

$f''(x) = (-2)^2 e^{-2x}$

$\vdots$

$f^{(n)}(x) = (-2)^n e^{-2x}$

$f^{(n)}(2) = (-2)^n e^{-2 \cdot 2}$

So $e^{-2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-2)^n e^{-4}}{n!}(x-2)^n}$

For radius of conv., use ratio test:

$$\lim_{n\to\infty} \left| \frac{2^{n+1} e^{-4} (x-2)^{n+1}}{(n+1)!} \right| \cdot \left| \frac{n!}{2^n e^{-4} (x-2)^n} \right| = \lim_{n\to\infty} \frac{2|x-2|}{n+1} = 0$$

So conv. for all x, so radius of conv. $= \infty$

4. (3 points) Find a power series representation for the function $f(x)$ which satisfies

$$f'(x) = \frac{\cos(x^2) - 1}{x^3}, \qquad f(0) = 10.$$

$$\frac{\cos(x^2)-1}{x^3} = \frac{\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right) - 1}{x^3}$$

$$= \frac{-\frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots}{x^3} = -\frac{x}{2!} + \frac{x^5}{4} - \frac{x^9}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{x^{4n+1}}{[2(n+1)]!}$$

So $f(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)!} x^{4n+1} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{(4n+2)(2n+2)!} + C$

since $f(0) = 10$, $C = 10$

So $$\boxed{f(x) = 10 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(4n+2)(2n+2)!} x^{4n+2}}$$

5. (3 points) Consider the curve defined parametrically by

$$x = t^2, \quad y = t^3 - 3t$$

Find all times t where the tangent line is inclined at an angle of 45° (that is, makes an angle of 45° with the x-axis).

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 3$$

slope $= 1 \Rightarrow \frac{dx}{dt} = \frac{dy}{dt}$

$$2t = 3t^2 - 3$$

$$0 = 3t^2 - 2t - 3$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4+36}}{6} = \frac{2 \pm \sqrt{40}}{6}$$

$$= \frac{1 \pm \sqrt{10}}{3}$$

$(\approx 1.387$ and $-.721)$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Table of Integration Formulas Constants of integration have been omitted.

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \dfrac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \dfrac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$

*19. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$

*20. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$