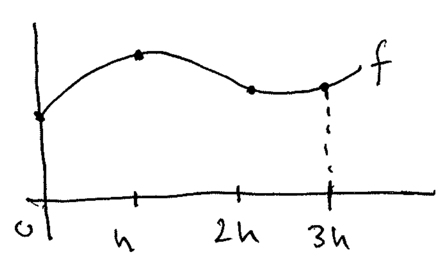
Cubic version of Simpson's Rule

(Also known as Simpson's 3/8 rule; see Wikipedia article for Simpson's rule.)

Goal: Develop a rule for approximating integrals $\int_a^b f(x)\,dx$ by approximating f by a number of cubic curves (degree 3 polynomials).

Step 1: Approximation using 1 cubic curve.

Let $f(x)$ be a function defined on interval $[0, 3h]$ for $h > 0$. We want to approximate $\int_0^{3h} f(x)\,dx$ using the values of f at $0, h, 2h, 3h$.



So our approximation takes the form

$$\int_0^{3h} f(x)\,dx \simeq A f(0) + B f(h) + C f(2h) + D f(3h)$$

for some constants A, B, C, D which we need to determine.

Our idea is to approximate f using a degree 3 polynomial. In particular, if $f(x)$ is a poly. of degree ≤ 3, the above approximation should give the exact answer.

By taking $f(x) = x^3$, then $f(x) = x^2$, then $f(x) = x$, then $f(x) = 1$, this will give 4 equations that will allow us to find A, B, C, D. For example, taking $f(x) = x^3$ gives:

$$\int_0^{3h} x^3 = A \cdot 0^3 + B \cdot h^3 + C \cdot (2h)^3 + D \cdot (3h)^3$$

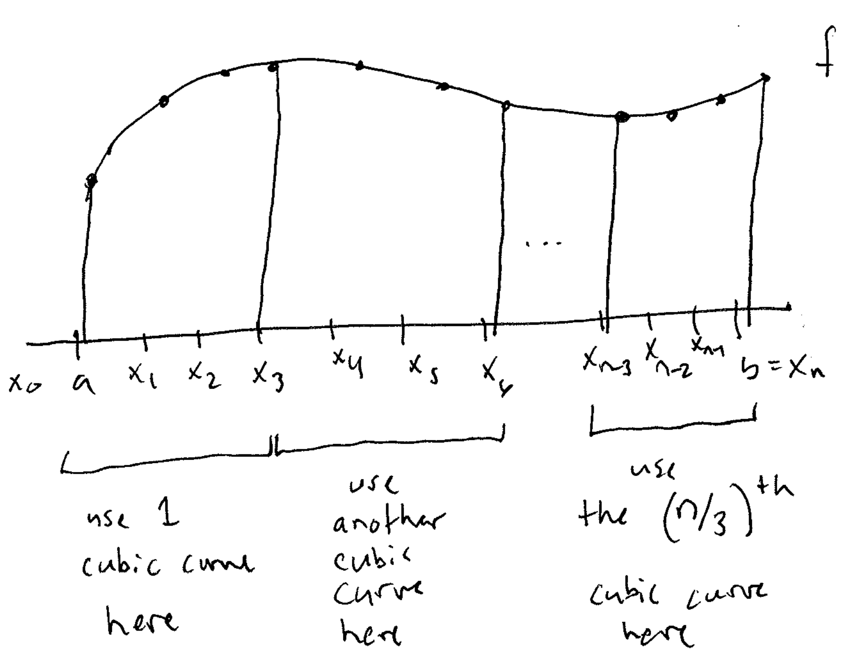
$$\frac{81}{4} h^4 = B h^3 + 8 C h^3 + 27 D h^3$$

$$\frac{81}{4} h = B + 8C + 27D$$

Write down the other 3 equations and then use all 4 equations to find A, B, C, D.

(The answer is $A = \frac{3h}{8}$, $B = \frac{9h}{8}$, $C = \frac{9h}{8}$, $D = \frac{3h}{8}$.)

<u>Step 2</u>: Now consider a function $f(x)$ defined on an interval $[a,b]$. Let n be positive integer divisible by 3. We want to approximate $\int_a^b f(x)\,dx$ by ~~breaking~~ using $n/3$ cubic curves to approximate f:



Use your answer from Step 1 to write down an approximation for $\int_a^b f(x)\,dx$ in terms of $\Delta x = \frac{b-a}{n}$ and $f(x_0), f(x_1), \ldots, f(x_n)$.

Application

Simpson's rules are used in naval architecture to find areas and volumes of irregular shapes.

Below is a cross-section of a lifeboat. Use your answer from Step 1 to approximate the area.