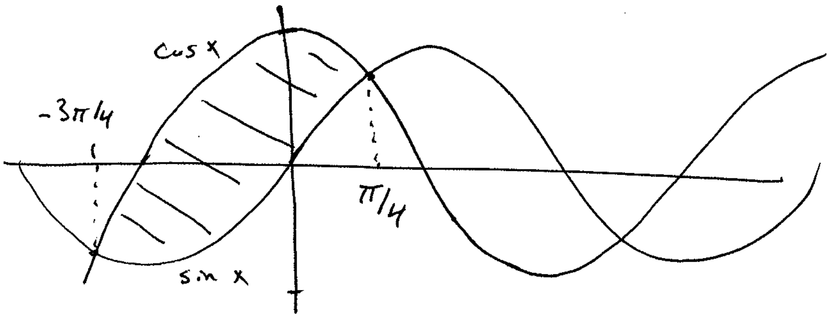
Math 2924-050
Fall 2014
Exam 1

Name:____

Problem	**Points**
Problem 1 (15 pts)	
Problem 2 (10 pts)	
Problem 3 (20 pts)	
Problem 4 (15 pts)	
Problem 5 (10 pts)	
Problem 6 (10 pts)	
Problem 7 (20 pts)	
Total	

1. (15 points) Find the area of the region bounded by the curves $y = \cos x$ and $y = \sin x$ with $-3\pi/4 \le x \le \pi/4$.



$$\text{Area} = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_{-3\pi/4}^{\pi/4}$$

$$= 2\sqrt{2}$$

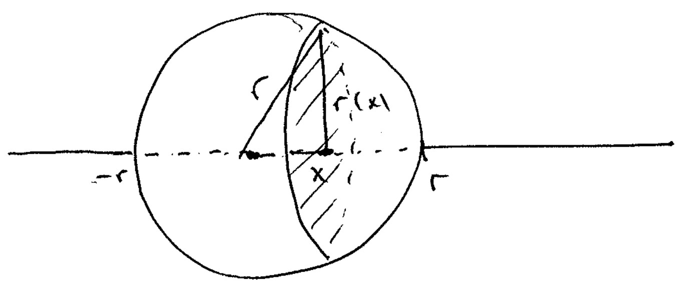
2. (10 points) Find the derivative of $y = x^{\cos x}$.

$$\ln y = \cos x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln x + \frac{\cos x}{x}$$

$$\frac{dy}{dx} = x^{\cos x}\left(-\sin x \cdot \ln x + \frac{\cos x}{x}\right)$$

3. (20 points) Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.



$$r(x) = \sqrt{r^2 - x^2}$$

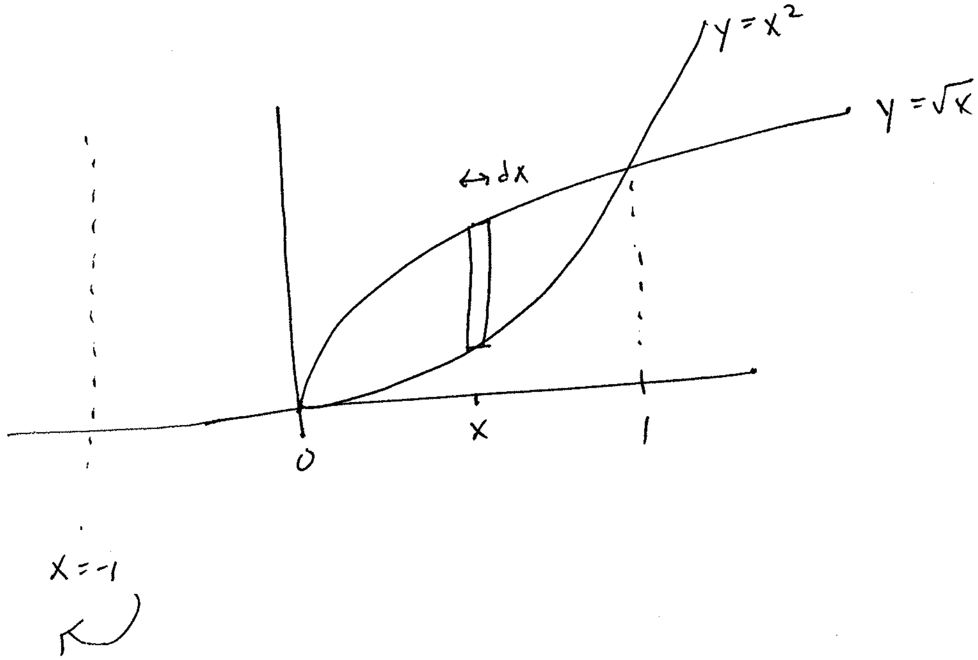
Area of cross-section $= \pi r(x)^2$

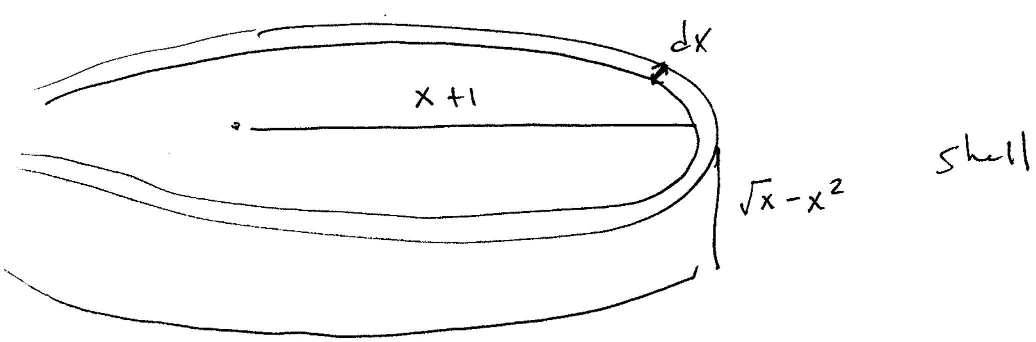
$= \pi (r^2 - x^2)$

$$\text{volume} = \int_{-r}^{r} \pi (r^2 - x^2)\, dx = \pi \left(r^2 x - \frac{1}{3} x^3 \right) \Big|_{-r}^{r}$$

$$= 2\pi \cdot \left(r^3 - \frac{1}{3} r^3 \right) = \frac{4}{3} \pi r^3$$

4. (15 points) Set up an integral to find the volume of the solid obtained by rotating the bounded region contained between the curves $y^2 = x$, $y = x^2$ about the line $x = -1$. (You do not need to evaluate the integral.)





volume of shell $= 2\pi (x+1)(\sqrt{x} - x^2)\, dx$

total volume $= \displaystyle\int_0^1 2\pi (x+1)(\sqrt{x} - x^2)\, dx$

5. (10 points) Newton's Law of Universal Gravitation says that two objects with masses m_1 and m_2 of distance r apart exert a gravitational force of magnitude

$$F = Gm_1m_2/r^2$$

on each other. G is a constant. Assuming the position of the first mass is fixed, how much work is required to move the second mass from a distance of $r = a$ to a distance of $r = b$ (you may assume $0 < a < b$)?

$$W = \int_a^b F \, dr = Gm_1m_2 \int_a^b \frac{1}{r^2} \, dr$$

$$= Gm_1m_2 \left(\frac{1}{a} - \frac{1}{b} \right)$$

6. (10 points) Let $y = f(x)$ and $x = g(y)$ be functions which are inverses of each other. Suppose $f'(x) = 1/x$. Prove that $g'(y) = g(y)$ by differentiating the identity $g(f(x)) = x$.

$$g(f(x)) = x$$

$$\Rightarrow \quad g'(f(x))\, f'(x) = 1 \quad \text{(chain rule)}$$

$$g'(y) \cdot \frac{1}{x} = 1$$

$$g'(y) = x = g(y)$$

7. (20 points) Evaluate the integrals.

a) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln|u| + C$

$u = \sin x$

$du = \cos x \, dx$

$= \ln|\sin x| + C$

b) $\int \frac{e^x}{2^x} \, dx = \int e^{x(1-\ln 2)} \, dx = \frac{1}{1-\ln 2} e^{x(1-\ln 2)} + C$