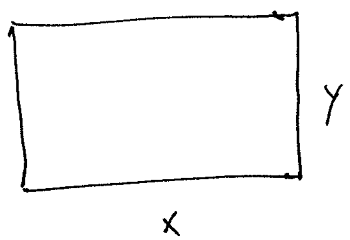
Problem 1

Show that among all rectangles with a fixed perimeter P, the square with sides of length $\frac{P}{4}$ maximizes area.



Constraint: $P = 2x + 2y$

Maximize: $A = xy$

Solution:

$$y = \frac{P}{2} - x$$

$$A = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$A' = \frac{P}{2} - 2x = 0$$

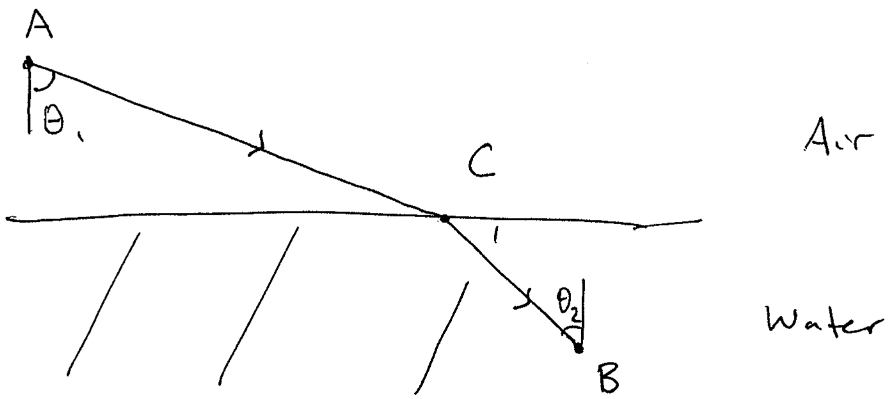
$$x = \frac{P}{4}$$

$$y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

So $x = y = \frac{P}{4}$ is unique crit. pt. of A.

Since $A'' = -2$, A is concave down everywhere, hence this crit. pt. is global max.

Problem 2

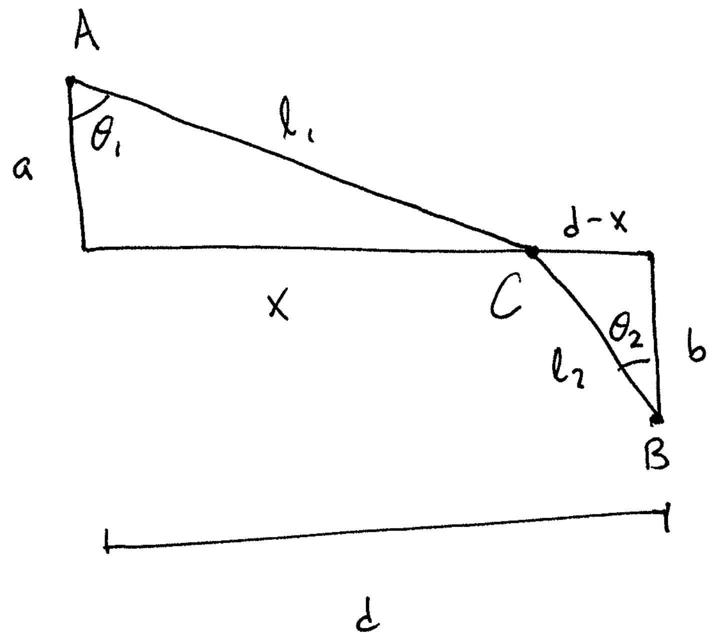


A light ray refracts (changes its path) when it passes from one medium to another, for example from air to water.

This can be explained by the following principle:

The speed of light in air is v_1, the speed of light in water is v_2, and the path from A to B that the light ray takes is the path that minimizes the time taken.

Show that $\dfrac{\sin \theta_1}{\sin \theta_2} = \dfrac{v_1}{v_2}$.



a, b, d are constants (depending on the positions of A, B).

x is a variable which represents the position of C, $0 \le x \le d$.

l_1 = distance between A, C, l_2 = distance between C, B.

Let $T(x)$ = time for light to travel from A to C to B.

Then
$$T(x) = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}.$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} + \frac{-(d-x)}{v_2\sqrt{b^2+(d-x)^2}} = \frac{x}{v_1 l_1} + \frac{x-d}{v_2 l_2} = \frac{x v_2 l_2 + (x-d) v_1 l_1}{v_2 l_2 v_1 l_1}$$

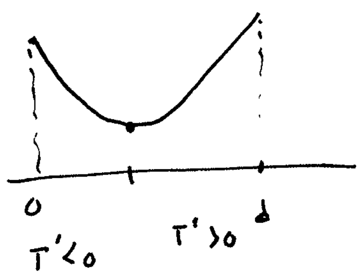
$$= \frac{x(v_1 l_1 + v_2 l_2) - d v_1 l_1}{v_2 l_2 v_1 l_1}$$

So $T'(x) = 0 \iff x = \dfrac{d v_1 l_1}{v_1 l_1 + v_2 l_2}$

Also, $T'(0) = \dfrac{-dv_1\ell_1}{v_2\ell_2 v_1\ell_1} = -\dfrac{d}{v_2\ell_2} < 0$

$$T'(d) = \frac{dv_1\ell_1 + dv_2\ell_2 - dv_1\ell_1}{v_2\ell_2 v_1 \ell_1} = \frac{d}{v_1\ell_1} > 0$$

So the critical point is a local min by first derivative test, and actually is global min for $0 \le x \le d$ by above inequalities



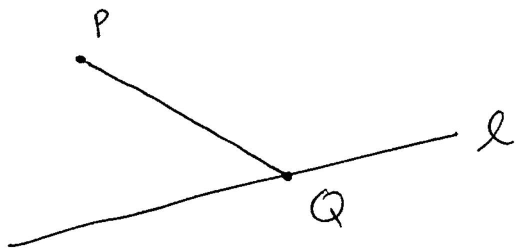
So the position of C is given by $x = \dfrac{dv_1\ell_1}{v_1\ell_1 + v_2\ell_2}$.

Now we calculate:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\frac{x}{\ell_1}}{\frac{d-x}{\ell_2}} = \frac{x\ell_2}{\ell_1(d-x)} = \frac{x\ell_2}{\ell_1\left(d - \frac{dv_1\ell_1}{v_1\ell_1+v_2\ell_2}\right)}$$

$$= \frac{x\ell_2}{\ell_1\left[\frac{d(v_1\ell_1+v_2\ell_2) - dv_1\ell_1}{v_1\ell_1+v_2\ell_2}\right]} = \frac{x\ell_2}{\ell_1\left[\frac{dv_2\ell_2}{v_1\ell_1+v_2\ell_2}\right]} = \frac{dv_1\ell_1}{v_1\ell_1+v_2\ell_2}\cdot\frac{\ell_2}{\ell_1}\cdot\frac{v_1\ell_1+v_2\ell_2}{dv_2\ell_2} = \frac{v_1}{v_2}.$$

Problem 3



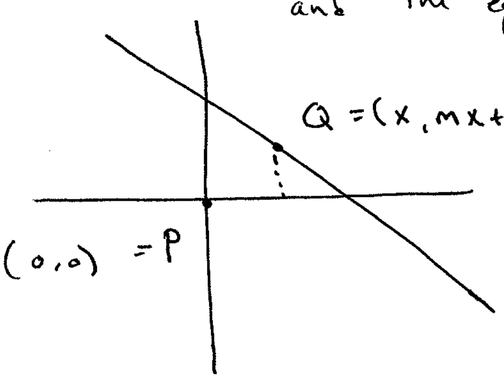
Given a line ℓ and a point P not on ℓ, let $d = d(P,Q)$ be the distance between P and Q. Show that d is minimized when $\overline{PQ} \perp \ell$. (P is fixed and Q can vary on ℓ.)

Hints:

1) d is minimized $\Leftrightarrow$ d^2 is minimized

2) lines with slopes m_1 and m_2 are perpendicular $\Leftrightarrow$ $m_1 m_2 = -1$

3) Without loss of generality you may assume $P = (0,0)$ and the eqtn of ℓ is $y = mx + b$ with $b \neq 0$.



$$d^2 = x^2 + (mx+b)^2$$

$$(d^2)' = 2x + 2(mx+b)m = 0$$

$$x(1+m^2) = -mb$$

$$x = -\frac{mb}{1+m^2}$$

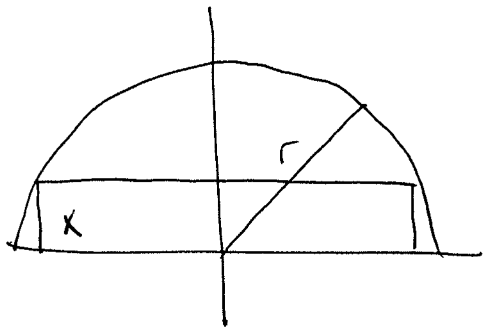
(since $\lim_{x \to \pm\infty} d^2 = \infty$, this crit # is absolute min)

$$\text{slope of } \overline{PQ} = \frac{m\left(-\frac{mb}{1+m^2}\right) + b}{-\frac{mb}{1+m^2}} = -\frac{1+m^2}{mb} \cdot \frac{b}{1+m^2} = -\frac{1}{m}$$

Thus $\overline{PQ}$ is perpendicular to the line $y = mx + b$

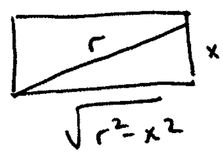
Problem 4

Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



Let x = height of rectangle, $0 \le x \le r$.

Then width $= 2\sqrt{r^2 - x^2}$



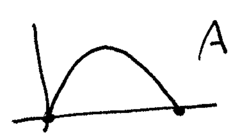
So area $= A = 2x\sqrt{r^2 - x^2}$

$$A' = 2\sqrt{r^2-x^2} + 2x \cdot \frac{-2x}{2\sqrt{r^2-x^2}}$$

$$= \frac{2(r^2-x^2)}{\sqrt{r^2-x^2}} - \frac{2x^2}{\sqrt{r^2-x^2}} = \frac{2r^2-4x^2}{\sqrt{r^2-x^2}}$$

So $A' = 0 \iff 2r^2 - 4x^2 = 0 \iff \frac{1}{2}r^2 = x^2 \Rightarrow x = \frac{r}{\sqrt{2}}$

Since $A'(0) > 0$, $A'(r) < 0$, A looks like



So $x = \frac{r}{\sqrt{2}}$ is the global max. Hence max area is $A\left(\frac{r}{\sqrt{2}}\right) = r^2$.