SOLUTIONS

Calculus and Analytic Geometry 1, Math 1823-001, Fall 2014
Practice Exam 1

1. Find the following limits exactly. Write DNE if they do not exist. Allow $\infty, -\infty$ as possible answers.

a) $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

$\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\frac{1}{t}\left(1-\frac{1}{t+1}\right)=\frac{1}{t}\left(\frac{t+1-1}{t+1}\right)=\frac{1}{t+1}$

so $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{1}{t+1}=1.$

b) $\lim_{h\to 3}\frac{h^2+2}{h+3}$

$=\frac{3^2+2}{3+3}=\frac{11}{6}$

c) $\lim_{x\to -3}\left(\frac{x^2-9}{x+3}\right)^{1/3}=\lim_{x\to -3}\left[\frac{(x-3)(x+3)}{x+3}\right]^{1/3}=\lim_{x\to -3}(x-3)^{1/3}=(-3-3)^{1/3}$

$=(-6)^{1/3}$

d) $\lim_{y\to -1}(\cos(y+1)\sin(\pi y/2))$

$=\cos(-1+1)\cdot\sin(-\pi/2)=1\cdot(-1)=-1$

e) $\lim_{x\to 0}\sin^2(1/x)$

DNE

oscillates between 0 and 1 as $x\to 0$

f) $\lim_{x\to 1}\frac{1/x-1}{x-1}$

$\frac{1/x-1}{x-1}=\frac{\frac{1-x}{x}}{x-1}=\frac{1-x}{x}\cdot\frac{1}{x-1}=-\frac{1}{x}$

so $\lim_{x\to 1}\frac{1/x-1}{x-1}=\lim_{x\to 1}\left(-\frac{1}{x}\right)=-1$

g) $\lim_{x\to\sqrt{2}}(2x^2-3x+5)$

$=2(\sqrt{2})^2-3\sqrt{2}+5=4-3\sqrt{2}+5=9-3\sqrt{2}$

h) $\lim_{x\to 1^-}\frac{2(x-1)}{|x-1|}$

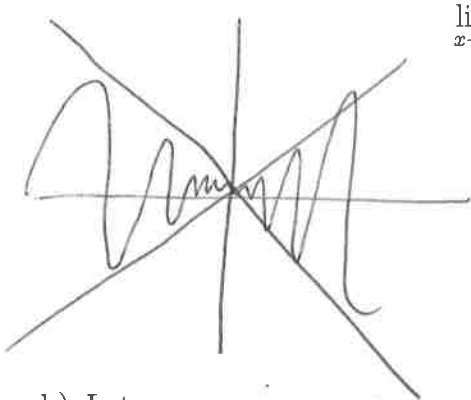
for $x<1$, $\frac{x-1}{|x-1|}=-1$, so $\lim_{x\to 1^-}\frac{2(x-1)}{|x-1|}=-2$

i) $\lim_{t\to 0^+}\frac{1}{\sin t}=\infty$

because $\sin t\to 0^+$ as $t\to 0^+$, so $\frac{1}{\sin t}=\frac{1}{\text{small pos. \#}}=$ large pos. # for $t$ small pos. #

2. a) Explain with a picture why

$$\lim_{x\to 0}\left[x\sin(1/x)\right] = 0.$$



So $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

b) Let

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Explain why $f'(0)$ does not exist.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \text{ DNE}$$

c) Let

$$g(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that $g'(0) = 0$.

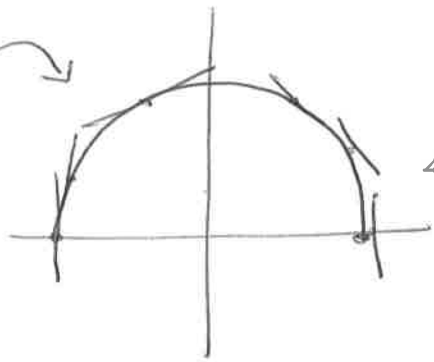
$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{h^2\sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0 \text{ by part a).}$$

3. Let $f : [-1, 1] \to \mathbb{R}$ be the function $f(x) = \sqrt{1 - x^2}$. The graph of $f$ is the upper half of a circle of radius 1. Draw the graph and use it to explain why

$$\lim_{a \to -1^+} f'(a) = \infty, \qquad \lim_{a \to 1^-} f'(a) = -\infty.$$

slopes of tangent lines $\longrightarrow \infty$ as $a \to -1^+$



slopes of tangent lines $\longrightarrow -\infty$ as $a \longrightarrow 1^-$

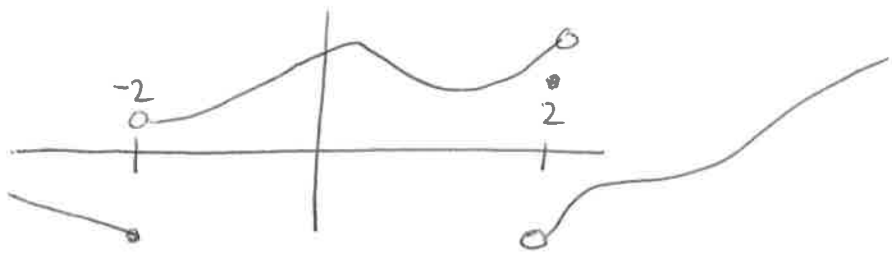
4. Explain the geometric significance of each of the following limits:

$$\underbrace{\lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}}, \qquad \underbrace{\lim_{x \to a} \frac{x^2 - a^2}{x - a}}.$$

this is slope of tangent line to graph of $f(x) = \sqrt{x}$ at $a$

slope of tangent line to graph of $f(x) = x^2$ at $a$.

5. Draw a picture of a function that is continuous everywhere except at $x = -2$ and $x = 2$, and, furthermore, is continuous from the left at $-2$ and not continuous from the right or left at 2.




6. Let

$$f(x) = \begin{cases} 2x^2 + ax & x \leq 1, \\ -3x + 2 & x > 1. \end{cases}$$

Find the number $a$ so that $f$ is continuous everywhere.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(2x^2 + ax\right) = 2 + a$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-3x + 2) = -1$$

Need the limits to agree, so need $2 + a = -1$

thus $a = -3$

7. Let $P_1(x)$ and $P_2(x)$ be polynomials. Let $Q(x) = P_1(x)/P_2(x)$. At which points is $Q(x)$ discontinuous?

$Q(x)$ is discontinuous at the points

$a$ where $P_2(a) = 0$.

8. In the $\epsilon, \delta$ definition of a limit, the inequalities

$$|f(x) - L| < \epsilon, \qquad 0 < |x - a| < \delta$$

appear. What are the geometric meanings of these inequalities?

$|f(x)-L| < \varepsilon$ means the distance between $f(x)$ and $L$ is less than $\varepsilon$

$0 < |x-a| < \delta$ means the distance between $x$ and $a$ is less than $\delta$ and $x$ is not equal to $a$

9. a) Let $f(x) = -x^2 + 1$. For $a$ an arbitrary number, find $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{-(a+h)^2 + 1 - (-a^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-a^2} - 2ah - h^2 \cancel{+1} \cancel{+a^2} \cancel{-1}}{h} = \lim_{h \to 0} (-2a - h) = -2a.$$

b) Find the equation of the line tangent to the graph of $f$ at $x = 3$.

line has slope $f'(3) = -2 \cdot 3 = -6$

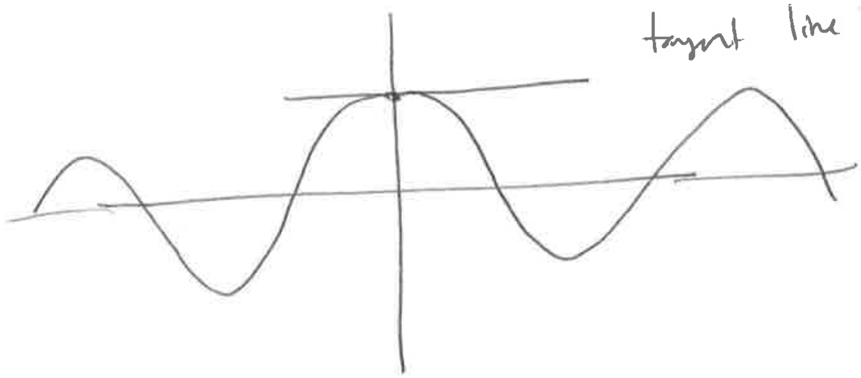
goes through point $(3, f(3)) = (3, -8)$.

So equation is

$$y - (-8) = -6(x-3), \text{ or}$$

$$y = -6x + 10$$

10. Let $h(t) = \cos t$. Draw the graph of $h$, and use the graph to find $h'(0)$.



tangent line is horizontal at 0,

so $h'(0) = 0$

11. Suppose $f(x)$ and $g(x)$ are continuous functions on all of $\mathbb{R}$, and

$$f(2) = 4, \quad g(2) = 3, \quad g(1) = 2.$$

Find the following limits:

a) $\lim_{x \to 2} [2f(x) - g(x)]$ $= 2 \cdot 4 - 3 = 5$

b) $\lim_{x \to 1} (f \circ g)(x)$ $= (f \circ g)(1) = f(g(1)) = f(2) = 4$

c) $\lim_{x \to 2} \sqrt{f(x)}$ $= \sqrt{f(2)} = \sqrt{4} = 2$

d) $\lim_{x \to 2} \sin\left(\frac{f(x)}{g(x)}\right)$ $= \sin\left(\frac{f(2)}{g(2)}\right) = \sin\left(\frac{4}{3}\right).$