Homework 2, Topics in Topology, Fall 2017

1. (a) Suppose X is a full subcomplex of $\mathbb{R}^3$ with its standard integer cubing. Show that X is NPC if and only if no 3-cube in $\mathbb{R}^3$ has exactly seven of its vertices in X.

(b)* Let $D = \{(n,n,n) \mid n \in \mathbb{Z}\}$ and let X be the combinatorial k-neighborhood of D (for some integer $k \geq 1$). That is, X is the full subcomplex of $\mathbb{R}^3$ whose vertex set is all $(x,y,z) \in \mathbb{Z}^3$ at most k edges away from D. Show that X is NPC (in fact, CAT(0)).

(c)** Show that the analogous subcomplex of $\mathbb{R}^4$ is not NPC.

2. Let X be a CAT(0) cube complex. Prove that if g is an automorphism of X and H is a half-space of X such that $gH \subset H$ and $\partial H \cap \partial gH = \emptyset$, then g does not fix any vertex of X.

3. (a) Suppose X and Y are flag simplicial complexes and Z is a full subcomplex of both. Show that $X \cup_Z Y$ is a flag simplicial complex.

(b) Let A, B, C be NPC cube complexes, and suppose

$$\phi_A \colon C \to A, \quad \phi_B \colon C \to B$$

are combinatorial isometric embeddings of C into A and B. Let

$$X \ = \ A \sqcup B / \phi_A(c) = \phi_B(c) \ \ \forall c \in C$$

(i.e. glue A and B along their embedded copies of C). Show that X is an NPC cube complex.

4. Prove that, given three points a, b, c in a metric space, there exists a comparison tripod with valence one vertices a', b', c' such that $d(a,b) = d(a',b')$, $d(b,c) = d(b',c')$ and $d(c,a) = d(c',a')$.

5. (a) Let Y be the three-point set $\{a,b,c\}$ and let W be the set of all partitions of Y into two subsets. Then (Y,W) is a space with walls. Find the CAT(0) cube complex associated with (Y,W).

(b) Same as (a), but with the four-point set $\{a,b,c,d\}$. Which vertices are principal ultrafilters? Note that there is a maximal cube, all of whose vertices are non-principal.

Remark: there is a notion of *median algebra* which is a set with a ternary operation $m(x,y,z)$ satisfying some axioms. Any median space is a median algebra. If you do the construction above with an n point set, you get the "free median algebra" on n generators. The case $n = 5$ is already quite large and complicated.