Homework 1, Topics in Topology, Fall 2017

1. Prove the equivalence of the following statements about a group G:
(a) G is residually finite
(b) for any finite set $F \subset G$, there is a finite quotient $\phi\colon G \to Q$ such that $\phi|_F$ is injective
(c) $\bigcap_{H \dot{<} G} H = \{1\}$
(d) the profinite topology on G is Hausdorff

2. Prove the equivalence of the following statements about a group G and a subgroup $H < G$:
(a) H is separable in G
(b) $\bigcap_{H < K \dot{<} G} K = H$
(c) H is closed in the profinite topology

3. Suppose $G_0 \dot{<} G$ and $H < G$. Let $H_0 = H \cap G_0$. Prove that the following are equivalent:
(a) H_0 is separable in G_0
(b) H_0 is separable in G
(c) H is separable in G

4. Show that every graph with finitely generated fundamental group has a compact core. [Def: a *compact core* of a space X is a compact subspace $C \subset X$ such that the inclusion map $C \hookrightarrow X$ is a homotopy equivalence. In the case of a graph, this is just a finite subgraph whose inclusion induces an isomorphism on fundamental groups.]

5. Show that every closed surface Σ with $\chi(\Sigma) < 0$ is a finite-sheeted cover of Σ_{-1}, the non-orientable closed surface with Euler charcteristic -1. [You may want to rely on the classification of surfaces.]

6. Prove that the group $\mathbb{Z}^n$ is LERF. (Do it for $n = 2$ if you prefer.)