You may apply theorems from the course, but please give the name or statement of the theorem.

1. Consider the long exact sequence of homotopy groups

\[ \cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots \]

for a triple \((X, A, B)\). Show that the sequence is exact at the \(\pi_n(X, B, x_0)\) term.

First, \(\text{Im}(i_*) \subset \text{Ker}(j_*)\): recall that an element \([f] \in \pi_n(X, A, x_0)\) is trivial if and only if \(f\) is homotopic (through maps of triples) to a map with image in \(A\). So \(j_*(i_*([g])) = [j \circ i \circ g]\) is trivial for any \([g] \in \pi_n(A, B, x_0)\) since \(j \circ i \circ g\) already has image in \(A\).

Next, \(\text{Ker}(j_*) \subset \text{Im}(i_*)\): if \([f] \in \text{Ker}(j_*)\) then after a homotopy we can assume that \(f\) maps \((I^n, \partial I^n, J^{n-1})\) into \((A, B, x_0)\), and now it represents an element of \(i_*(\pi_n(A, B, x_0))\).

2. (a) State the Lefschetz fixed point theorem, and define the Lefschetz number \(\tau(f)\) of a map \(f: X \to X\), where \(X\) is a (retract of a) finite simplicial complex. (Note that this includes compact CW complexes.)

(b) Give the homology groups and cohomology groups of \(\mathbb{C}P^n\) in \(\mathbb{Z}\)-coefficients, and also describe the cup product structure (no proof needed).

(a) If \(X\) is a retract of a finite simplicial complex then its homology groups are finitely generated, and non-zero in only finitely many dimensions. Define the Lefschetz number \(\tau(f)\) to be \(\sum (-1)^i \text{tr}(f_*: H_i(X)/T_i \to H_i(X)/T_i)\), where \(H_i(X)/T_i\) is the homology group with torsion factored out. The Lefschetz fixed point theorem says that if \(\tau(f) \neq 0\) then \(f\) has a fixed point.

(b) \(H_i(\mathbb{C}P^n)\) is \(\mathbb{Z}\) for \(i\) even and \(0 \leq i \leq 2n\) and \(0\) otherwise. The cohomology groups \(H^i(\mathbb{C}P^n; \mathbb{Z})\) are the same. If \(\alpha\) is a generator of \(H^2(\mathbb{C}P^n; \mathbb{Z})\) then \(\alpha^3\) generates \(H^2(\mathbb{C}P^n; \mathbb{Z})\) for \(i \leq n\). That is, \(H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})\) with \(|\alpha| = 2\).

(c) If \(f^*: H^i(\mathbb{C}P^n) \to H^i(\mathbb{C}P^n)\) is multiplication by \(d\), what is the map \(f_*: H_i(\mathbb{C}P^n) \to H_i(\mathbb{C}P^n)\)? Explain why, using the universal coefficient theorem.

(d) Prove that every map \(f: \mathbb{C}P^{2k} \to \mathbb{C}P^{2k}\) has a fixed point. [Hint: use the cup product.]

(e) For every \(i\) the group \(H_{i-1}(\mathbb{C}P^n)\) is free, so \(\text{Ext}(H_{i-1}(\mathbb{C}P^n), \mathbb{Z}) = 0\), and hence \(H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \text{Hom}(H_i(\mathbb{C}P^n), \mathbb{Z})\) by the universal coefficient theorem. Moreover, by naturality we have the commutative diagram

\[
\begin{array}{ccc}
H^i(\mathbb{C}P^n; \mathbb{Z}) & \xrightarrow{h \cong} & \text{Hom}(H_i(\mathbb{C}P^n), \mathbb{Z}) \\
\uparrow d & & \uparrow (f_*)^* \\
H^i(\mathbb{C}P^n; \mathbb{Z}) & \xrightarrow{h \cong} & \text{Hom}(H_i(\mathbb{C}P^n), \mathbb{Z})
\end{array}
\]

and therefore \((f_*)^*\) is multiplication by \(d\). This implies that \(f_*\) is also multiplication by \(d\).
(d) Let \( n = 2k \). Let \( \alpha \in H^2(\mathbb{C}P^n; \mathbb{Z}) \) be a generator and suppose \( f^*(\alpha) = d\alpha \). Then \( f^*(\alpha^i) = (f^*(\alpha))^i = d^i\alpha^i \), so \( f^* : H^2(\mathbb{C}P^n; \mathbb{Z}) \to H^2(\mathbb{C}P^n; \mathbb{Z}) \) is multiplication by \( d^i \). By part (c) the map \( f : H_2(\mathbb{C}P^n) \to H_2(\mathbb{C}P^n) \) is also multiplication by \( d^i \). So \( \tau(f) = \sum_{i=0}^{n} (-1)^{2i + 1} d^i = \sum_{i=0}^{n} d^i \). Now \( d^0 = 1 \) and \( \sum_{i=1}^{n} d^i \) is even, so \( \tau(f) \neq 0 \).

3. Prove the extension lemma: Let \((X, A)\) be a finite-dimensional CW pair and \(Y\) a path connected space such that \( \pi_{n-1}(Y) = 0 \) for all \( n \) for which \( X - A \) has an \( n \)-cell. Then every map \( f : A \to Y \) can be extended to a map \( X \to Y \). [Hint: define the extension cell-by-cell, in increasing dimensions.]

First, extend to the 0-cells of \( X - A \) by sending them to any points of \( Y \). Next, assume that \( f \) has been extended to \( X^{(n-1)} \cup A \). Let \( e \) be an \( n \)-cell of \( X - A \) with attaching map \( \phi : \partial D^n \to X^{(n-1)} \). Then \([f \circ \phi] \in \pi_{n-1}(Y)\) and this group is trivial by assumption. So \( f \circ \phi \) extends to a map \( D^n \to Y \), and this extension joins with \( f \) to give an extension to \( (X^{(n-1)} \cup e) \cup A \). The other \( n \)-cells can be handled simultaneously, since their interiors are disjoint. By induction on \( n \), \( f \) extends to all of \( X \).

4. Let \((X, A)\) be a CW pair with \( A \) contractible. Use the extension lemma to show that \( A \) is a retract of \( X \).

Consider the identity map \( i : A \to A \), and note that \( \pi_{n-1}(A) = 0 \) for all \( n \). Hence, by the extension lemma, \( i \) extends to a map \( r : X \to A \). (We must either assume \( X \) finite-dimensional, or that the extension lemma holds without this assumption (it does).) Now, \( r \) is a retraction since it is the identity on \( A \).

5. Let \( K \subset S^3 \) be a knot, i.e. an embedded circle. Let \( N \) be a closed \( \varepsilon \)-neighborhood of \( K \) which is homeomorphic to the solid torus \( D^2 \times S^1 \). Let \( X = S^3 - \text{int}(N) \). Note that \( N \) and \( X \) are both compact 3-manifolds with boundary, with common boundary \( X \cap N \), which is a torus.

Use the Mayer-Vietoris sequence to compute the first homology group of \( X \). [It turns out the answer does not depend on whether \( K \) is actually knotted or not!]

We have the following portion of the Mayer-Vietoris sequence for \( S^3 \) expressed as the union of \( N \) and \( X \):

\[
\rightarrow H_2(S^3) \to H_1(N \cap X) \to H_1(N) \oplus H_1(X) \to H_1(S^3) \to
\]

and note that \( H_2(S^3) = H_1(S^3) = 0 \). Hence \( H_1(N \cap X) \cong H_1(N) \oplus H_1(X) \). Next, \( H_1(N \cap X) \cong H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \) and \( H_1(N) \cong \mathbb{Z} \) since \( N \) deformation retracts onto a circle. Then we have \( H_1(X) \cong H_1(N \cap X)/H_1(N) \cong (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \) for some \( n \). Since \( H_1(N) \oplus H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \), we must have \( n = 1 \) and \( H_1(X) \cong \mathbb{Z} \).

For \( H_2(X) \) we can proceed similarly, but the Mayer-Vietoris sequence gives slightly less information. Namely, the sequence

\[
\rightarrow H_3(N) \oplus H_3(X) \to H_3(S^3) \to H_2(N \cap X) \to H_2(N) \oplus H_2(X) \to H_2(S^3) \to
\]
yields the exact sequence

\[
0 \to \mathbb{Z} \to \mathbb{Z} \to H_2(X) \to 0
\]
since \( H_2(N) = H_2(S^3) = 0 \). (The groups \( H_3(N) \) and \( H_3(X) \) vanish since \( N \) and \( X \) are 3-manifolds with non-empty boundary.) Thus \( H_2(X) \) is finite cyclic.
In fact, $H_2(X) = 0$, as we can see using Lefschetz duality. First, $H_3(X, \partial X) \cong \mathbb{Z}$, and in the long exact sequence for the pair $(X, \partial X)$, the boundary map takes a fundamental class to a fundamental class of the boundary. Thus, the exact sequence

$$
\rightarrow H_3(X, \partial X) \rightarrow H_2(\partial X) \rightarrow H_2(X) \rightarrow H_2(X, \partial X)
$$

has the form

$$
\rightarrow \mathbb{Z} \cong \mathbb{Z}^{\oplus 0} \rightarrow H_2(X) \rightarrow H_2(X, \partial X).
$$

Hence $H_2(X)$ is a subgroup of $H_2(X, \partial X)$, and must be zero if $H_2(X, \partial X)$ has no torsion. Indeed, $H_2(X, \partial X) \cong H^1(X)$ by Lefschetz duality, and this latter group is $\text{Hom}(H_1(X), \mathbb{Z}) \cong \mathbb{Z}$.

**Remark:** If $K$ is unknotted then $X$ is homeomorphic to the solid torus $D^2 \times S^1$. The calculations above show that $X$ has the same homology as $D^2 \times S^1$, even if $K$ is knotted. So homology does not detect knottedness.